10.3 Parametric equations and calculus

Thm. For a smooth curve C where x=f(t) and y=g(t),

the slope at (x,y) is $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}},\ \dfrac{dx}{dt}\neq 0$

2nd Derivative:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

3rd Derivative:

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]=\frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{\frac{dx}{dt}}$$

ex. For $x=4-t^2$ and $y=t^2+4t$:
a. For what values of t are there vertical or horizontal tangent lines.?
b. Find the slope at (3,5).
c. Determine the concavity.

slope $= \frac{dy}{dx}$

$\frac{dx}{dt} = -2t \qquad \frac{dy}{dt} = 2t+4$

a

vertical when $-2t=0$, $t=0$

horizontal when $2t+4=0$, $t=-2$

b

$\frac{dy}{dx} = \frac{2t+4}{-2t}$

$3=4-t^2$, $5=t^2+4t$

$t^2=1$ $\qquad 0=t^2+4t-5$

$t=\pm 1$ $\qquad 0=(t+5)(t-1)$

$t=-5, 1$

use $t=1$

$\frac{dy}{dx} = \frac{2(1)+4}{-2(1)} = \frac{6}{-2} = -3$

$$\frac{d^2y}{dx^2} = \frac{\frac{2(-2t)+(2t+4)(+2)}{(-2t)^2}}{\frac{-2t}{1}}$$

$$= \frac{\cancel{-4t}+\cancel{4t}+8}{(-2t)^3}$$

$$= \frac{8}{-8t^3} = \frac{-1}{t^3}$$

for t in $(-\infty, 0)$
$\frac{d^2y}{dx^2} > 0$, so the curve is concave up.

for t in $(0, \infty)$
$\frac{d^2y}{dx^2} < 0$, so the curve is concave down.

For some object,
position: x=f(t) and y=g(t)

velocity: x=f '(t) and y=g '(t)

speed: $speed(t) = |v(t)|$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}$$

$$= \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$$

Preview: When these are written like ordered pairs, they can be interpreted as vectors.

Thm. For a smooth curve x=f(t) and y=g(t) that does not intersect itself for any t in [a,b] except possibly at endpoints, the arc length over the interval is

$$s=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_a^b \sqrt{\left(f'(t)\right)^2+\left(g'(t)\right)^2}\,dt$$

So, if v(t) is velocity,
the distance traveled along this curve is

$$=\int_a^b |v(t)|\,dt=\int_a^b speed(t)\,dt$$

Though in the book, surface area for parametric equations is NOT on the AP Calculus BC exam.

ex. Use the preceding theorem, show that the circumference of the circle: $x=4\cos\theta$ and $y=4\sin\theta$ for $0\leq\theta\leq 2\pi$ is 8π.

$$\text{arc length} = \int_0^{2\pi} \sqrt{(-4\sin\theta)^2+(4\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{16\sin^2\theta+16\cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{16(\underbrace{\sin^2\theta+\cos^2\theta}_{})}\, d\theta = \int_0^{2\pi} \sqrt{16\cdot 1}\, d\theta$$

$$= \int_0^{2\pi} 4\, d\theta = 4\theta\Big|_0^{2\pi} = 8\pi - 0 = 8\pi$$

ex. Set up an integral for the arc length of $x=t^3$ and $y=2t^2$ for t in [0,1], then evaluate on the calculator.

$$\int_0^1 \sqrt{(3t^2)^2+(4t)^2}\,dt$$

$$=2.259$$

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fnInt(√(9X^4+16X
²),X,0,1)
         2.259259259
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