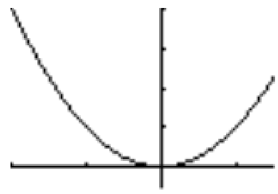
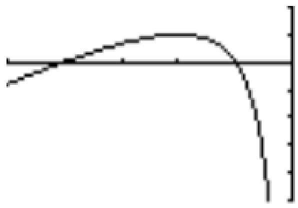
Do now as a warm-up:

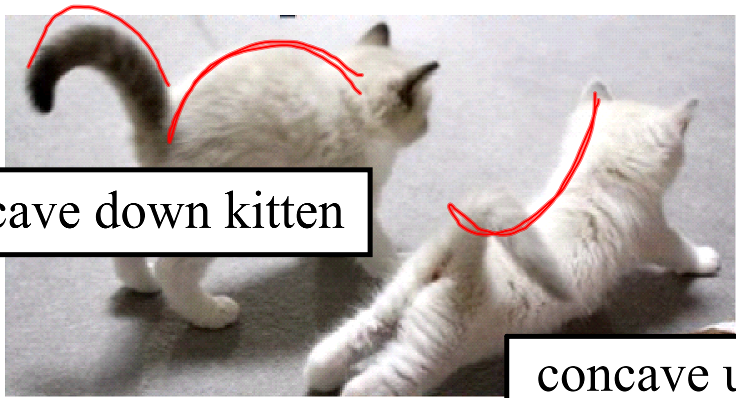
1. Suppose we drew many tangent lines for this first curve. How do the slopes of these tangent lines change as we look from left to right?



2. Suppose we drew many tangent lines for this second curve. How do the slopes of these tangent lines change as we look from left to right?



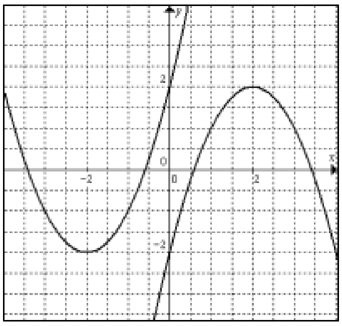
3.4 Concavity and the Second Derivative Test



concave down kitten

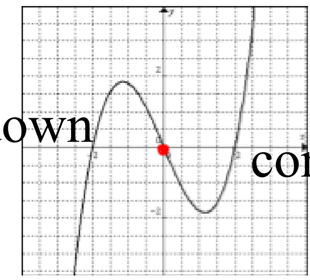
concave up kitten

concave up
(like a cup)



concave down
(like a frown)

concave down part



concave up part

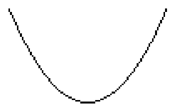
Defn. A point of inflection is a point where a graph changes from concave up to concave down or vice versa.

Thm. Test for Concavity

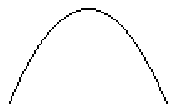
Let f be a function whose second derivative exists on an interval I, then

1. if $f''(x)>0$, then f is concave up.
2. if $f''(x)<0$, then f is concave down.

f'' +

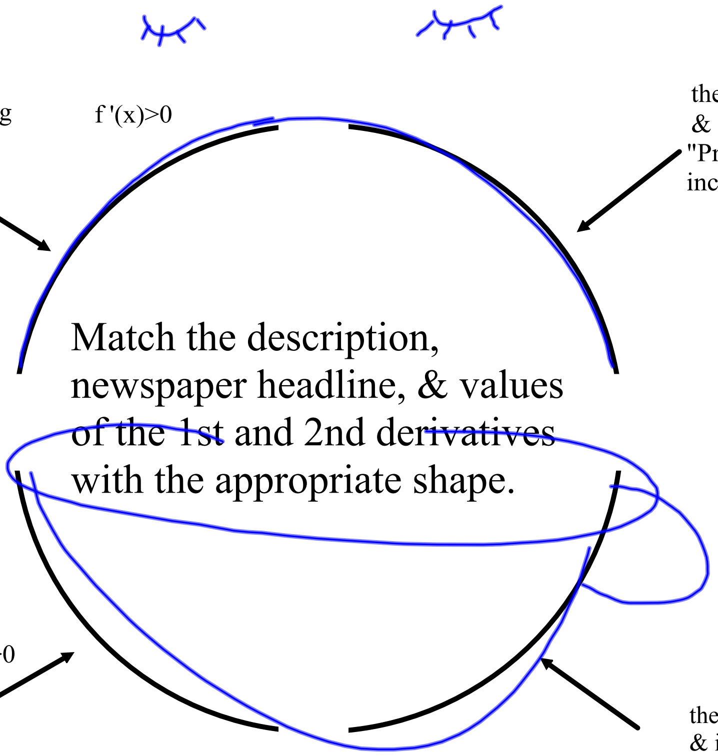


f'' -



the graph is increasing
& is concave down
Bank fees continue to rise,
ut are leveling off"
f "(x)<0
f '(x)>0

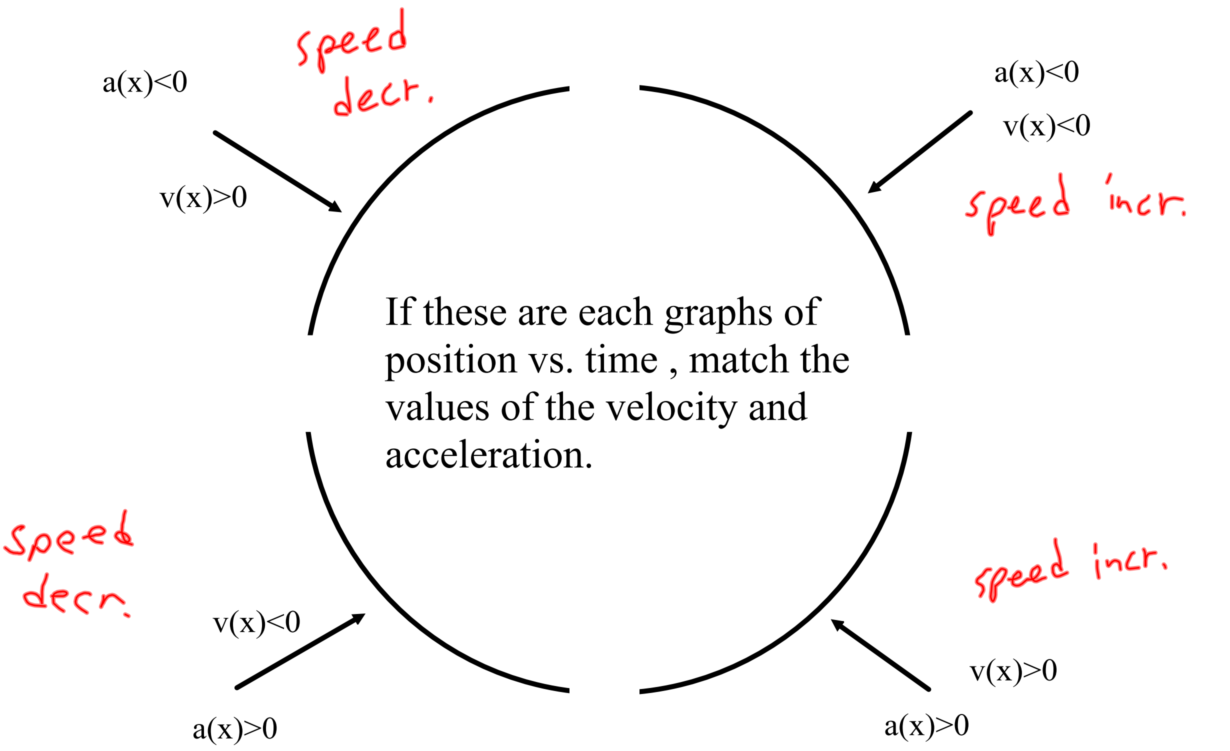
the graph is decreasing
& is concave down
"Prices are falling at an
increasing rate"
f "(x)<0
f '(x)<0



Match the description, newspaper headline, & values of the 1st and 2nd derivatives with the appropriate shape.

f '(x)<0
f "(x)>0
the graph is decreasing
& is concave up
"CD sales in 2003 dropped,
but rate of descent slowed"

f '(x)>0
the graph is increasing
& is concave up
"Data suggest that economic
recovery is picking up pace"
f "(x)>0



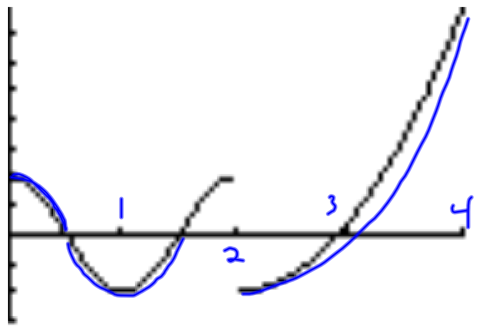
If these are each graphs of position vs. time , match the values of the velocity and acceleration.

When is the speed of the object increasing or decreasing?

Thm.

$$\begin{tikzcd}
\boxed{\begin{array}{l}(c, f(c)) \text{ is a} \\ \text{point of inflection}\end{array}} \arrow[r, "\text{always}"] & \boxed{\begin{array}{l} f''(c)=0 \\ \text{or} \\ f''(c) \text{ is undefined}\end{array}} \arrow[l, bend left, "\text{maybe}"]
\end{tikzcd}$$

ex. Function f(x) is graphed below and is defined on [0,4]. Estimate the intervals on which f '(x) is positive or negative and on which intervals f "(x) is positive or negative.



$f'(x) > 0$ $(1,2)$ & $(2,4)$

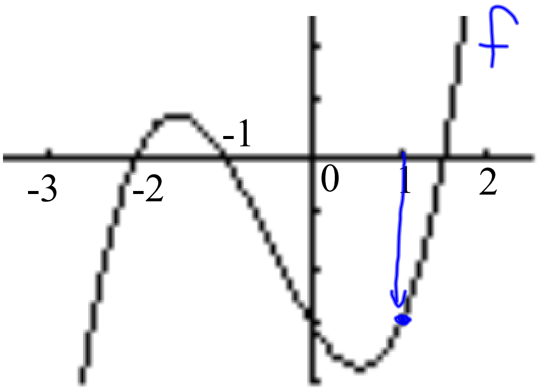
$f'(x) < 0$ $(0,1)$

$f''(x) > 0$ $(2,4)$
$(.5, 1.5)$

$f''(x) < 0$ $(0,.5)$ & $(1.5,2)$

ex. For the graph shown, at which integer value of x is it true that both f '(x)>0 and f ''(x)>0?

f incr. f is CU



at x = 1

ex. Determine the concavity and identify any points of inflection of

$$f(x) = \frac{2}{x} + \sqrt{x} = 2x^{-1} + x^{\frac{1}{2}}$$ domain $(0,\infty)$

$$f'(x) = -2x^{-2} + \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = 4x^{-3} - \frac{1}{4}x^{-\frac{3}{2}}$$

$$= \frac{4\cdot 4}{4\cdot x^3} - \frac{1}{4x^{3/2}}\cdot\frac{x^{3/2}}{x^{3/2}}$$

$$= \frac{16 - x^{3/2}}{4x^3}$$

$$4x^3 = 0$$

$$x = 0$$

not in f's domain

$$16 - x^{\frac{3}{2}} = 0$$

$$(16)^{\frac{2}{3}} = \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}}$$

$$\sqrt[3]{256} = x$$

$$4\sqrt[3]{4} = x$$

on $(0, 4\sqrt[3]{4})$, $f''(x) > 0$

so f is CU

on $(4\sqrt[3]{4}, \infty)$, $f''(x) < 0$

so f is CD

Since $f''(x)$ changes sign at $x = 4\sqrt[3]{4}$ and $f(4\sqrt[3]{4})$ exists there is a point of inflection there.

ex. Find the x coordinates of any points of inflection for the graph of

$$f(x) = 3x^5 - 5x^4$$

$f'(x) = 15x^4 - 20x^3$

$f''(x) = 60x^3 - 60x^2$

$60x^3 - 60x^2 = 0$

$60x^2(x-1) = 0$

$x = 0, \ x = 1$

on $(-\infty, 0)$ $f''(x) < 0$

on $(0, 1)$ $f''(x) < 0$

on $(1, \infty)$ $f''(x) > 0$

Since $f''(x)$ changes sign at $x=1$, $(1, f(1))$ is a p.o.i.

$(1, -2)$

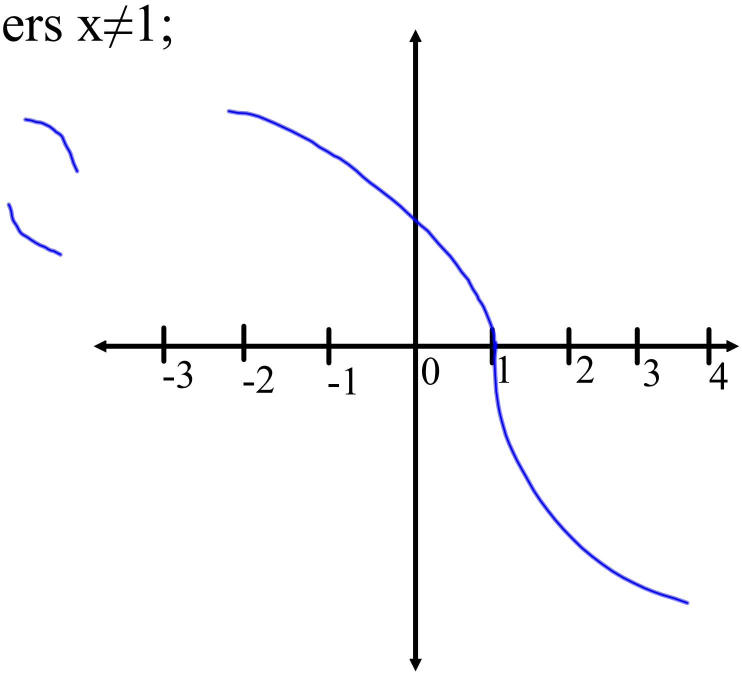
ex. Sketch the graph of a cont's function which satisfies all the following conditions:

$f'(x)<0$ for all real numbers $x\neq 1$;

$f'(1)$ does not exist;

$f''(x)<0$ for all $x<1$; and

$f''(x)>0$ for all $x>1$



Thm. Second Derivative Test (SDT)

f is continuous on I
f is differentiable on I, except possibly at c
c is a critical number
f ''(c) is +
is -

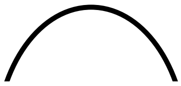
if f '' (c)=0, the SDT fails.

f(c) is a relative min
max

Review: First Derivative Test

f ' >0
on (_,c)

f '(c)=0

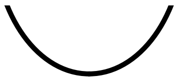


f ' <0
on (c,_)

f (c) is a max

---

f ' <0
on (_,c)



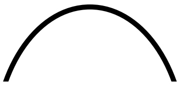
f '(c)=0

f ' >0
on (c,_)

f (c) is a min

New: Second Derivative Test

f '(c)=0



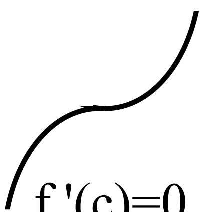
f "(c) <0

f (c) is a max

f '(c)=0

f "(c) >0

f (c) is a min



f '(c)=0

f "(c) =0

No conclusion
Second Derivative Test fails

Point of Inflection Test

$$\boxed{\begin{array}{l} f''(c) \text{ is 0 or undef'd} \\ f(c) \text{ is defined} \\ f'' \text{ changes sign at } x=c \end{array}} \longrightarrow \boxed{\begin{array}{l} (c,f(c)) \text{ is a} \\ \text{Point of Inflection} \end{array}}$$

ex. A function f is cont's on [-3,3] and its first and second derivatives are as follows:

x	(-3,-1)	-1	(-1,0)	0	(0,1)	1	(1,3)
$f'(x)$	Positive	0	Negative	Negative	Negative	0	Negative
$f''(x)$	Negative	Negative	Negative	0	Positive	0	Negative

critical #s

At what x values does f have...

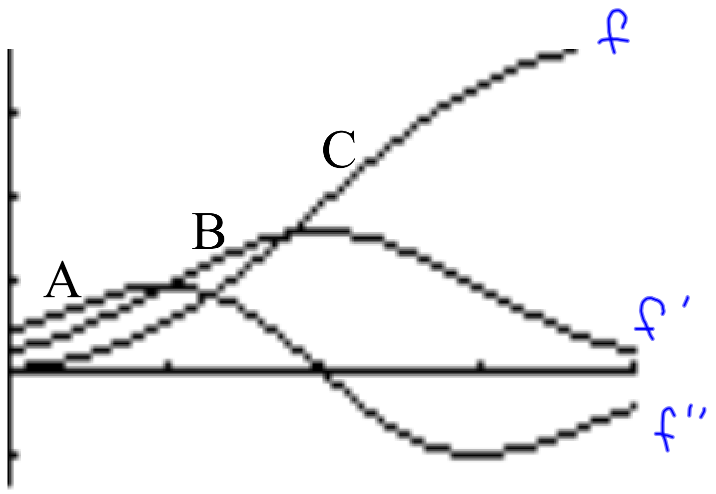
a. relative minima? Justify. nowhere, f' never changes from − to +.

b. relative maxima? Justify. at x=-1 f' changes + to − at x=-1 or $f''(-1)<0$

c. points of inflection? Justify.

at x=1 because f'' changes sign & (1, f(1)) exists since f is cont's on [-3,3].

ex. Which function is f, which is f ', and which is f ''?



Graph a function with the following properties:

$f(2)=f(4)=0$.

$f(3)$ is defined.

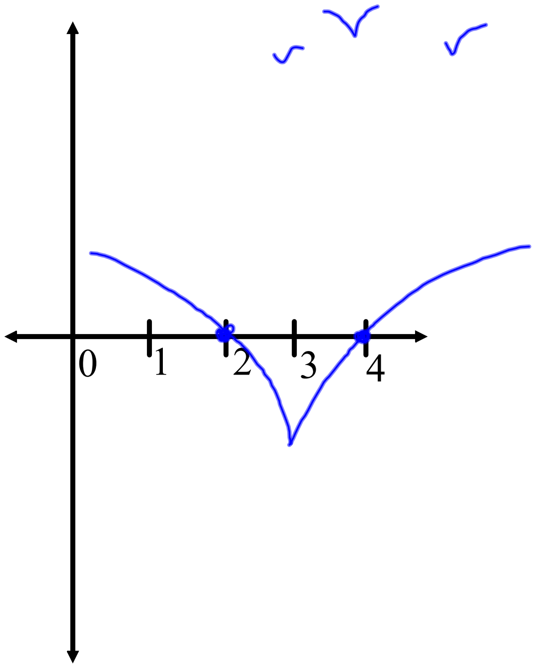
$f'(x)<0$ for $x<3$. ↘

$f'(3)$ is undefined. cusp or vertical slope

$f'(x)>0$ for $x>3$. ↗

$f''(x)<0$ for $x\neq3$.

CD



ex. The graph of f '(x) is shown.

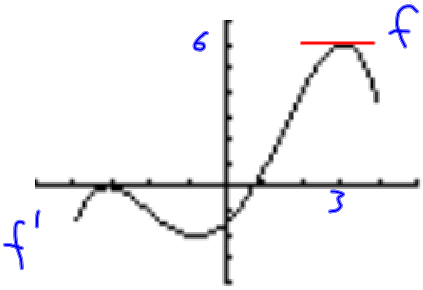
a. Suppose f(3)=1. Find the equation of the line tangent to f at (3,1). $y-1=6(x-3)$ $\quad 6=\frac{y-1}{x-3}$

b. Where does f have a local minimum? Justify. at $x=1$ since f' change from − to +

c. Estimate f ''(3). ≈ 0

d. Where does f have an inflection point? Justify.

$f'(3)=6$



if f' changes increasing to decreasing (or vice versa)

f'' changes sign.

So f has p.o.i. at

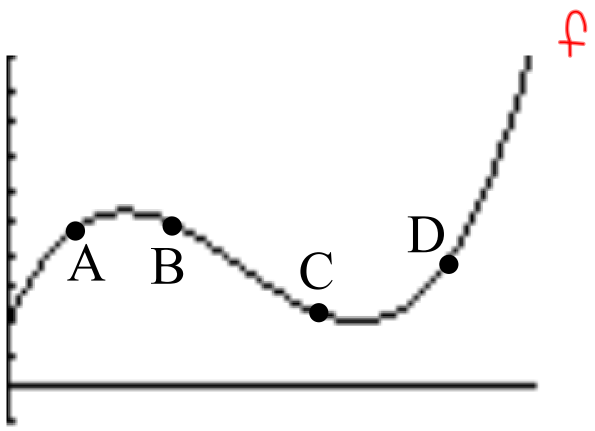
$x=-3, -1, 3$

ex. At which point(s) is the first derivative of f positive?

A & D incr.

At which point(s) is the second derivative of f positive?

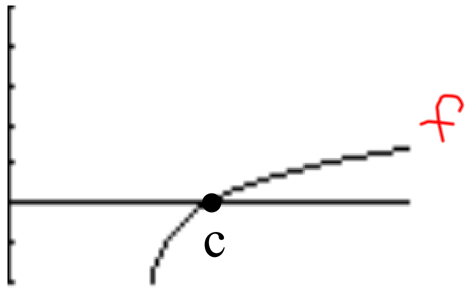
C & D CU



ex. Arrange these in order from least to greatest:

$f(c)=0$ $f'(c)>0$ $f''(c)<0$

root incr CD



$f''(c) < f(c) < f'(c)$

ex. The derivative of a function f is given by

$$f'(x) = (x-3)^2(x+2)$$

Where does f have points of inflection?

$$f''(x) = 2(x-3)^1(x+2) + (x-3)^2(1)$$

$$= (x-3)\left[\underbrace{2(x+2) + (x-3)}_{3x+1}\right]$$

$x-3=0$ $2x+4+x-3=0$

$x=3$ $3x=-1$

$x=-\frac{1}{3}$

on $(-\infty, -\frac{1}{3})$ $f''(x)>0$

on $(-\frac{1}{3}, 3)$ $f''(x)<0$

on $(3, \infty)$ $f''(x)>0$

POI occur at $x=-\frac{1}{3}$ & $x=3$.

because f'' changes sign at these x values.

ex. The derivative of a function f is given by

$$f(x) = x^4 - 4x^3 + 6x^2$$

Where does f have points of inflection?

$f'(x) = 4x^3 - 12x^2 + 12x$

$f''(x) = 12x^2 - 24x + 12$

$12x^2 - 24x + 12 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

possible poi at $x = 1$

There's no poi because f'' never changes sign.