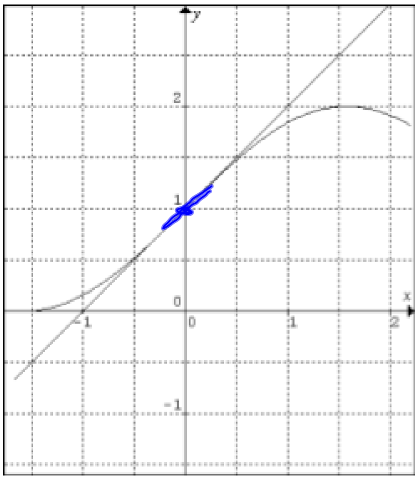
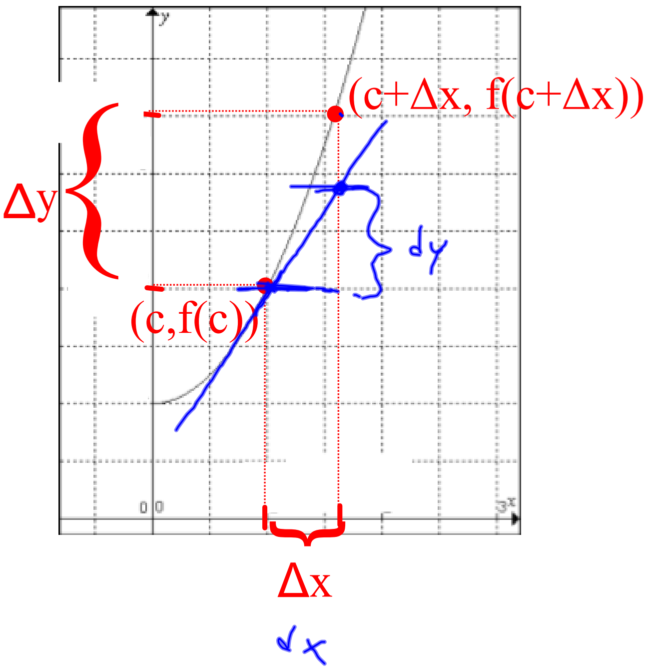
3.9 Differentials (linear approximations)

Use the tangent line to f(x)=1+sinx at x=0, y=x+1, to approximate f(0.5).



$f(.5) \approx .5 + 1 = 1.5$

The difference between two y values on the graph of a function is $\Delta y = f(c+\Delta x) - f(c)$.



That difference is close to $f'(c)\Delta x$.

Δx is often written dx. dx is called the differential of x.

Then $f'(c)\Delta x$ can be written $f'(c)dx$ and is called the differential of y.

Defn. Let $y=f(x)$ be differentiable, then the differential of x, written dx, is any non-zero number and the differential of y, written dy, is $dy=f'(x)dx$.

Note: $dy \approx \Delta y$

$dx = \Delta x$

Compare Δy to dy for the function $y=x^2$ when x=1 and dx=.01.

$y' = 2x$

c Δx

$dy = f'(c)dx$

$dy = 2(1)(.01)$

$= .02$

$\Delta y = f(c+\Delta x) - f(c)$

$= f(1+.01) - f(1)$

$= f(1.01) - f(1)$

$= 1.01^2 - 1^2$

$= 1.0201 - 1$

$= .0201$

What is the % error? $= \frac{.0201 - .02}{.0201} = \frac{.0001}{.0201} = \frac{1}{201}$

Error terms used in our book:

$$\underbrace{\Delta y}_{\text{propagated error}} = \overbrace{f(x+\underbrace{\Delta x}_{\text{measurement error}})}^{\text{exact value}} - \overbrace{f(x)}^{\text{measured value}}$$

ex. The radius of a sphere is measured to be 0.7 inch. The measurement could be off by as much as ±0.01 inch. Estimate the error in calculating the volume of the sphere.

$$V = \frac{4}{3}\pi r^3 \longrightarrow \frac{dV}{dr} = 4\pi r^2 dr$$

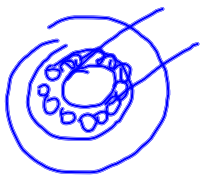
$$\Delta V \approx dV = 4\pi r^2 dr$$

actual error

$$dV = 4\pi(.7)^2(\pm .01)$$

$$= \pm .0196\pi$$

$$\approx \pm .061 \text{ or } .062 \text{ in}^3$$



ex. Find dy if y= $3x^2-4$

$y' = 6x$

$dy = y'\,dx = \underline{6x\,dx}$

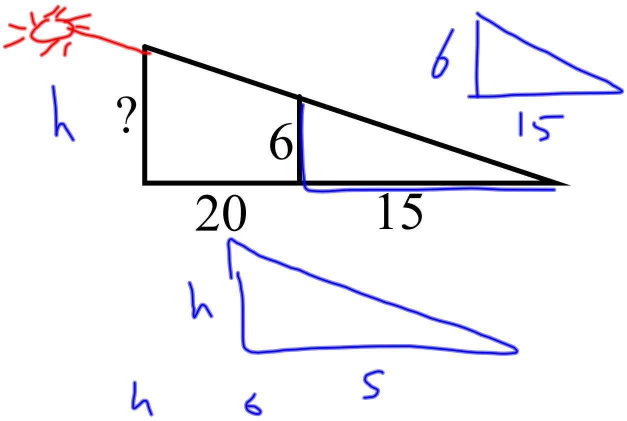
$\frac{dy}{dx} = y'$

ex. Find dy if y= $\tan^2 x = (\tan x)^2$

$y' = 2\tan x \cdot \sec^2 x$

$dy = 2\tan x \sec^2 x\,dx$

ex. To find the height of a lamppost, we stand a 6' pole 20' from the lamppost's base & measure the pole's shadow to be 15' long. If we believe our measurement could be off by as much as 1", what is our estimate of the lamppost's height?



$$\frac{h}{35} = \frac{6}{15}$$

$$h = \frac{6}{15} \cdot 35 = 14 \text{ ft.}$$

$$\frac{h}{s} = \frac{6}{15}$$

$$15h = 6s$$

$$h = \frac{6}{15} s$$

$$dh = \frac{6}{15} ds$$

$$= \frac{6}{15}\left(\frac{1}{12}\right) = \frac{1}{30} \text{ ft}$$

post is $14 \text{ ft} \pm \frac{1}{30}$ ft.

ex. Find the linear approximation of the function

$y = f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

at a=8 and use it to approximate the number $\sqrt[3]{7.99}$

approx $f(7.99)$

$f(8) = \sqrt[3]{8} = 2$

$\Delta x = dx = 8 - 7.99 = .01$

$y' = f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$dy = f'(x)\, dx$

$= \frac{1}{3\sqrt[3]{x^2}}\, dx$

$= \frac{1}{3\sqrt[3]{8^2}}(.01) = \frac{1}{3 \cdot 4}(.01)$

$= \frac{1}{12}(.01) = \frac{.01}{12} = \frac{1}{1200}$

$\sqrt[3]{7.99} \approx 2 - \frac{1}{1200} = 1.9991\bar{6}$

ex. Use differentials to approximate $\sqrt{24.8}$

$f(x) = x^{\frac{1}{2}}$

$x = 25$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Δx or $dx = -.2$

$dy = f'(x)\,dx = \frac{1}{2\sqrt{x}}\,dx$

$\frac{1}{2\sqrt{25}} \cdot (-.2) = \frac{-.2}{2.5} = -\frac{1}{50}$

$\sqrt{24.8} \approx \sqrt{25} - \frac{1}{50} = 5 - \frac{1}{50} = 4.98$