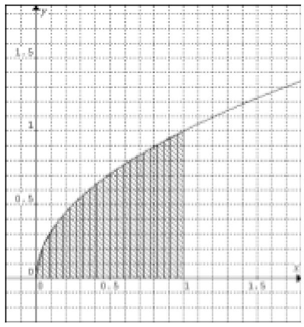
4.3 Riemann Sums and the Definite Integral

ex. Consider f(x)= $\sqrt{x}$ on [0,1].



For "nice" function values, let $x_i = \frac{i^2}{n^2}$

so $f(x_i) = \sqrt{\frac{i^2}{n^2}} = \frac{i}{n}$

then $\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{i^2 - (i^2-2i+1)}{n^2} = \frac{2i-1}{n^2}$

so area= $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(\Delta x) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)\left(\frac{2i-1}{n^2}\right)$

and we'll now find this limit. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)\left(\frac{2i-1}{n^2}\right)$

$$= \lim_{n\to\infty} \boxed{\sum_{i=1}^{n} \frac{2i^2 - i}{n^3}}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(2i^2 - i)$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=n}^{n} 2i^2 - \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[2\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i\right]$$

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FOIL

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{3 \cdot 2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{4n^3 + 6n^2 + 2n \;\; - 3n^2 - 3n}{6n^3}\right]$$

$$= \lim_{n\to\infty} \frac{4n^3}{6n^3} + \frac{3n^2}{6n^3} - \frac{n}{6n^3}$$

$$\frac{2}{3} + 0 - 0 = \frac{2}{3} = \text{area}$$

Defn.
Let f be defined on [a,b].

Partition [a,b] into n subintervals by the numbers x_i (where i=1, 2, 3, ..., n) so that $a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$

In each subinterval, choose any value c_i and let Δx_i be the length of that i^{th} subinterval.

Then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a Riemann Sum.

Defn.

If f is defined on [a,b] and $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x$ exists,

then f is "integrable" on [a,b] and that limit

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx$$

"upper limit of integration"

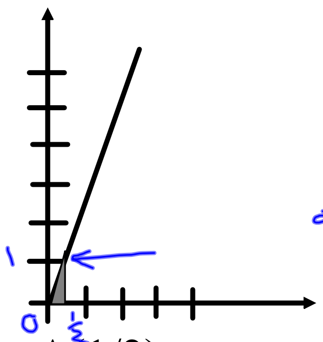
"lower limit of integration"

is called the "definite integral".

Note: $\Delta x \to 0$, means $n \to \infty$.

ex. Evaluate the following by using geometry area formulas:

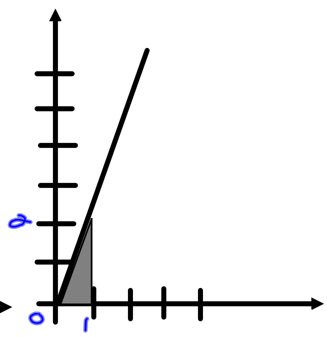
$$\int_0^{\frac{1}{2}} 2x\,dx$$



A(1/2)=

$\frac{1}{2}\left(\frac{1}{2}\right)1 = \frac{1}{4}$

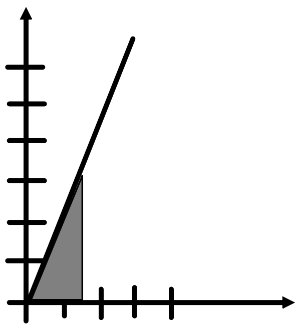
$$\int_0^{1} 2x\,dx$$



A(1)=

$\frac{1}{2}(1)(2) = 1$

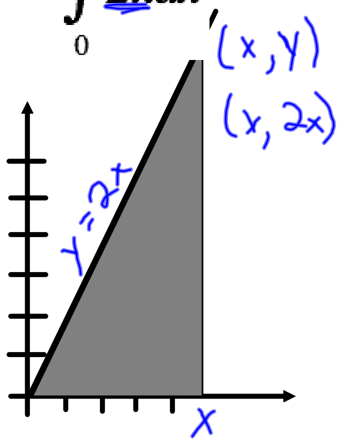
$$\int_0^{\frac{3}{2}} 2x\,dx$$



A(3/2)=

$\frac{1}{2}\left(\frac{3}{2}\right)(3) = \frac{9}{4}$

$$\int_0^{x} 2x\,dx$$



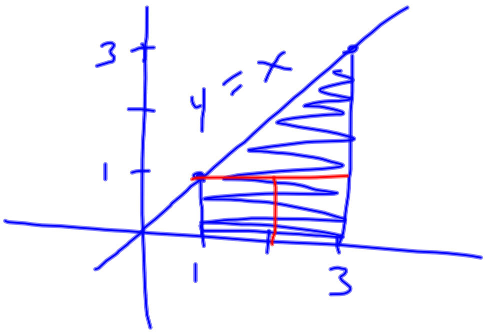
A(x)=

$\frac{1}{2}(x)(2x) = x^2$

Calc 3

$$\int_0^1 \boxed{\int_0^2 2xy \, dx} \, dy$$

ex. $\int_1^3 x \, dx$ Evaluate this using area formulas.



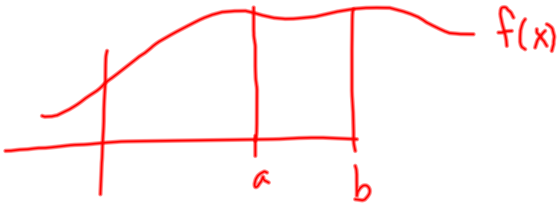
$1 \times 1 + 1 \times 1 + \frac{1}{2}(2)(2) = 4$

$= \frac{1}{2}(1+3)(2) = 4$

$\frac{1}{2}(b_1 + b_2)h$

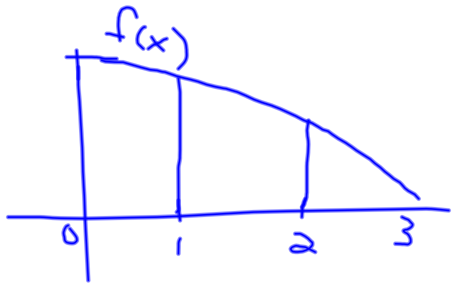
Defn. Two special integrals:
If f(x) is defined at x=a, then

1. $\int_a^a f(x)\,dx = 0$



2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$\int_1^2 f(x)\,dx = -\int_{2:00}^{1:00} f(x)\,dx$$



Properties of integrals (5 theorems)

If f is integrable on [a,b], $a<c<b$, $k\in R$...

...then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

...then $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

...then $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

...& f is non-negative, then $0 \le \int_a^b f(x)\,dx$

...& $f(x) \le g(x)\ \forall x \in [a,b]$

then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

ex. If $\int_0^5 f(x)\,dx=10$, $\int_5^7 f(x)\,dx=3$ and, $\int_3^5 f(x)dx = 2$ then...

a. $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx$

$= 10 + 3 = 13$

b. $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

c. $\int_5^5 f(x)\,dx = 0$

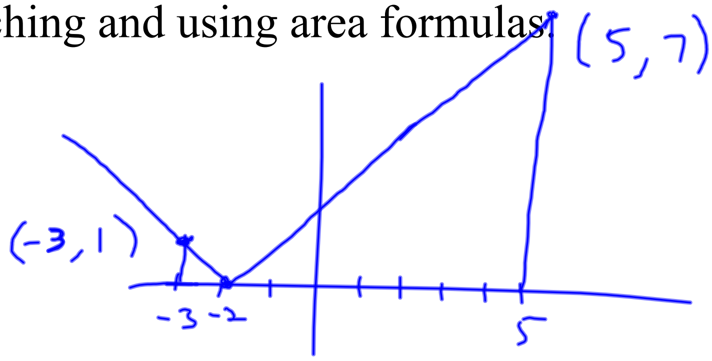
d. $\int_0^5 3f(x)\,dx = 3(10) = 30$

e. $\int_0^3 f(x)dx = \int_0^5 f(x)\,dx - \int_3^5 f(x)\,dx$

$= 10 - 2 = 8$

Evaluate by sketching and using area formulas.

$$\int_{-3}^{5} |x+2|\,dx$$



$$= \int_{-3}^{-2} |x+2|\,dx + \int_{-2}^{5} |x+2|\,dx$$

$$= \frac{1}{2}(1)(1) + \frac{1}{2}(7)(7)$$

$$\frac{1}{2} + \frac{49}{2} = 25$$

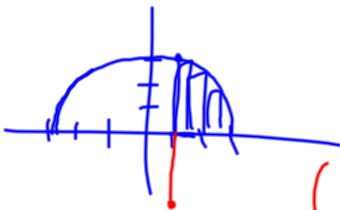
Evaluate the following by using area formulas or geometric properties:

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{\pi}{2}(3)^2 = \frac{9\pi}{2}$$

$y = \sqrt{9-x^2}$

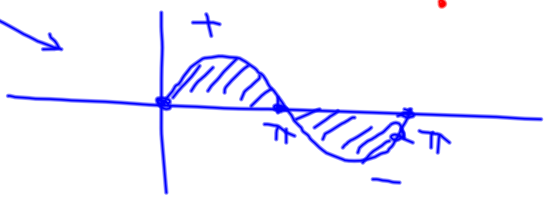
$y^2 = 9 - x^2$

$x^2 + y^2 = 9$



$$\int_{-3}^{3} \sqrt{9-x^2} + \int_{-3}^{3} -\sqrt{9-x^2}$$

$$\int_{0}^{2\pi} \sin x\,dx = 0$$



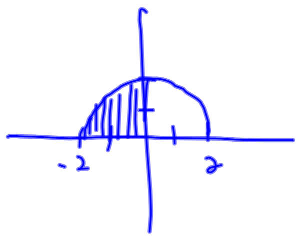
$\frac{b-a}{n}$ $\frac{a-b}{n}$

If $f(t)=\sqrt{4-t^2}$ find F(-2), F(0), and F(2) for $F(x)=\int_{-2}^{x} f(t)dt$

$F(-2) = \int_{-2}^{-2} f(t)dt = 0$

$F(0) = \int_{-2}^{0} f(t)\,dt = \frac{1}{4}\pi(2)^2$

$= \pi$



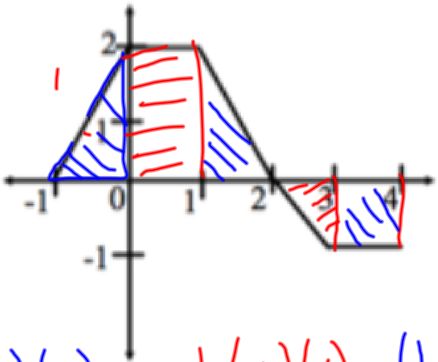
$F(2) = \int_{-2}^{2} f(t)dt = \frac{1}{2}\pi(2)^2 = 2\pi$

The graph of a piecewise-linear function f, on [-1,4] is shown.

upper limit

$$\int_{-1}^{4} f(x)dx =$$

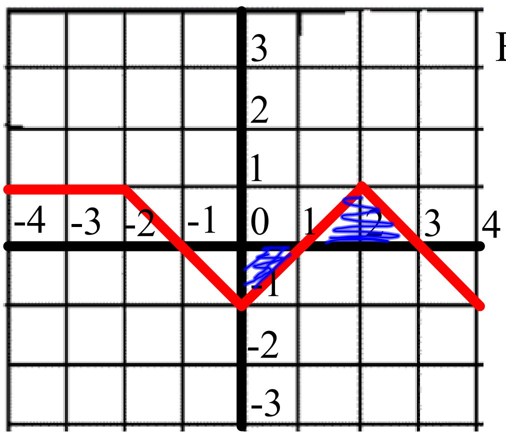
lower limit



$$= \frac{1}{2}(1)(2) + (1)(2) + \frac{1}{2}(1)(2) - \frac{1}{2}(1)(1) - (1)(1)$$

$$= 1 + 2 + 1 - \frac{1}{2} - 1$$

$$= 2\frac{1}{2} \text{ or } \frac{5}{2}$$



Evaluate the following:

$$\int_{-4}^{-2} f(x)dx = 2$$

$$\int_{-2}^{-4} f(x)dx = -2$$

$$\int_{-2}^{1} f(x)dx = -\frac{1}{2}$$

$$\int_{-4}^{4} f(x)dx = 2\frac{1}{2} - 1 + 1 - \frac{1}{2} = 2$$

$$\int_{0}^{3} f(x)dx = \frac{1}{2}$$