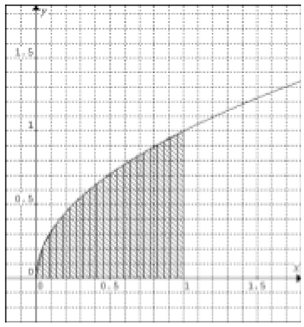
4.3 Riemann Sums and the Definite Integral

ex. Consider $f(x)=\sqrt{x}$ on [0,1].



For "nice" function values, let $x_i=\frac{i^2}{n^2}$

so $f(x_i)=\sqrt{\frac{i^2}{n^2}}=\frac{i}{n}$

then $\Delta x_i=\frac{i^2}{n^2}-\frac{(i-1)^2}{n^2}=\frac{i^2-(i^2-2i+1)}{n^2}=\frac{2i-1}{n^2}$

so area= $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)(\Delta x)=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)\left(\frac{2i-1}{n^2}\right)$

and we'll now find this limit. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)\left(\frac{2i-1}{n^2}\right)$

Defn.

Let f be defined on [a,b].

Partition [a,b] into n subintervals by the numbers x_i (where i=1, 2, 3, ..., n) so that $a=x_0<x_1<x_2<\dots<x_{n-1}<x_n=b$

In each subinterval, choose any value c_i and let Δx_i be the length of that i^{th} subinterval.

Then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a Riemann Sum.

Defn.

If f is defined on [a,b] and $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x$ exists,

then f is "integrable" on [a,b] and that limit

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx$$

b: "upper limit of integration"

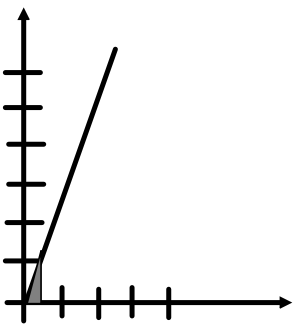
a: "lower limit of integration"

is called the "definite integral".

Note: $\Delta x \to 0$, means $n \to \infty$.

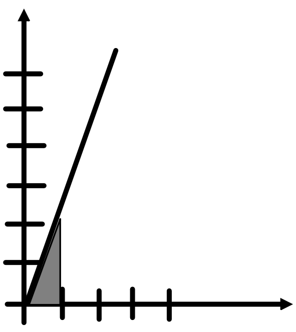
ex. Evaluate the following by using geometry area formulas:

$\int_0^{\frac{1}{2}} 2x\,dx$



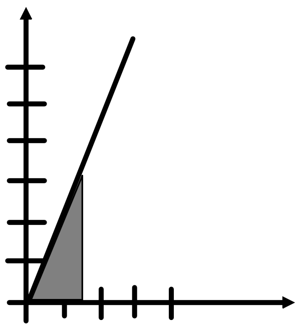
A(1/2)=

$\int_0^{1} 2x\,dx$



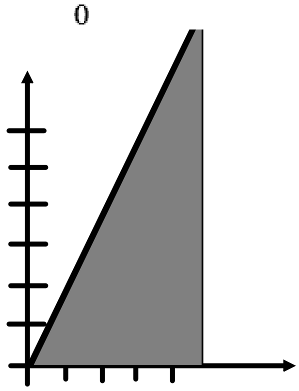
A(1)=

$\int_0^{\frac{3}{2}} 2x\,dx$



A(3/2)=

$\int_0^{x} 2x\,dx$

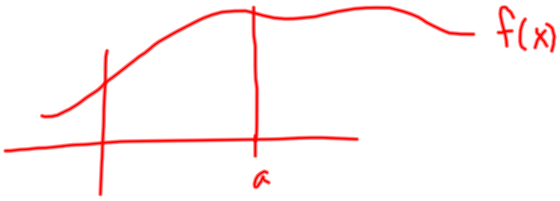


A(x)=

ex. $\int_1^3 x \, dx$ Evaluate this using area formulas.

Defn. Two special integrals:
If f(x) is defined at x=a, then

1. $\int_a^a f(x)\,dx = 0$



2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Properties of integrals (5 theorems)

If f is integrable on [a,b], a<c<b, $k \in R$...

...then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

...then $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

...then $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

...& f is non-negative, then $0 \leq \int_a^b f(x)\,dx$

...& $f(x) \leq g(x)\ \forall x \in [a,b]$
then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

ex. If $\int_0^5 f(x)\,dx=10$, $\int_5^7 f(x)\,dx=3$ and, $\int_3^5 f(x)dx = 2$ then...

a. $\int_0^7 f(x)\,dx$

b. $\int_5^0 f(x)\,dx$

c. $\int_5^5 f(x)\,dx$

d. $\int_0^5 3f(x)\,dx$

e. $\int_0^3 f(x)dx$

Evaluate by sketching and using area formulas.

$$\int_{-3}^{5} |x+2| \, dx$$

Evaluate the following by using area formulas or geometric properties:

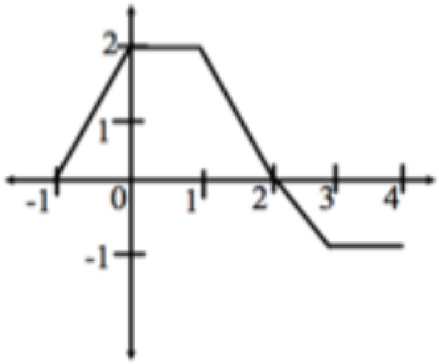
$$\int_{-3}^{3} \sqrt{9 - x^2}\,dx$$

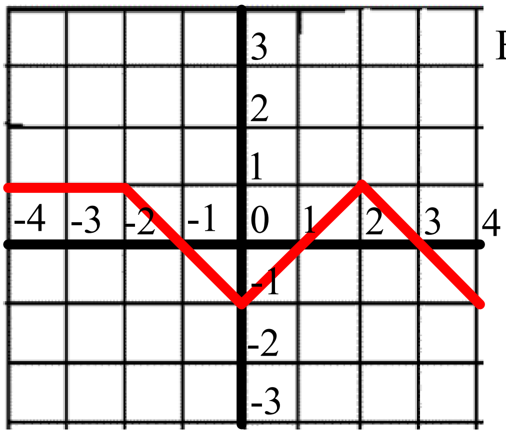
$$\int_{0}^{2\pi} \sin x\,dx$$

If $f(t)=\sqrt{4-t^2}$ find F(-2), F(0), and F(2) for $F(x)=\int_{-2}^{x} f(t)dt$

The graph of a piecewise-linear function f, on [-1,4] is shown.

$$\int_{-1}^{4} f(x)dx =$$





Evaluate the following:

$$\int_{-4}^{-2} f(x)dx$$

$$\int_{-2}^{-4} f(x)dx$$

$$\int_{-2}^{1} f(x)dx$$

$$\int_{-4}^{4} f(x)dx$$

$$\int_{0}^{3} f(x)dx$$