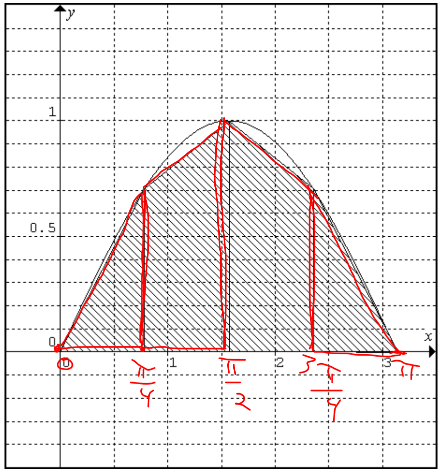
4.6 Numerical Integration

First Method: The Trapezoidal Rule

ex. Approximate $\int_0^{\pi} \sin x \, dx$ using n=4 and again with n=8.



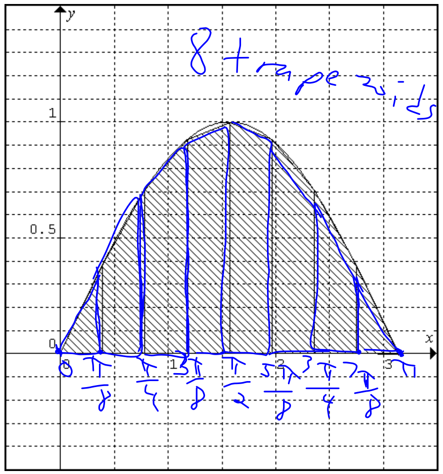
$$\frac{\pi}{4}\cdot\frac{1}{2}\left(\sin 0+\sin\frac{\pi}{4}\right)+\frac{\pi}{4}\cdot\frac{1}{2}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{2}\right)+\dots$$

$$=\frac{\pi}{4}\cdot\frac{1}{2}\left[\left(0+\frac{\sqrt2}{2}\right)+\left(\frac{\sqrt2}{2}+1\right)+\left(1+\frac{\sqrt2}{2}\right)+\left(\frac{\sqrt2}{2}+0\right)\right]$$

$$=\frac{\pi}{8}\left[2+4\left(\frac{\sqrt2}{2}\right)\right]=\frac{\pi}{8}\left[2+2\sqrt2\right]$$

$$=\frac{\pi(2+2\sqrt2)}{8}=\frac{\pi(1+\sqrt2)}{4}$$

$$\approx 1.8961$$



$$\frac{\pi}{8}\cdot\frac{1}{2}\left[0+(.3827)2+2(.7071)+2(.9239)+2(1)+2(.9239)+2(.7071)+2(.3827)+0\right]$$

$$\approx 1.9743$$

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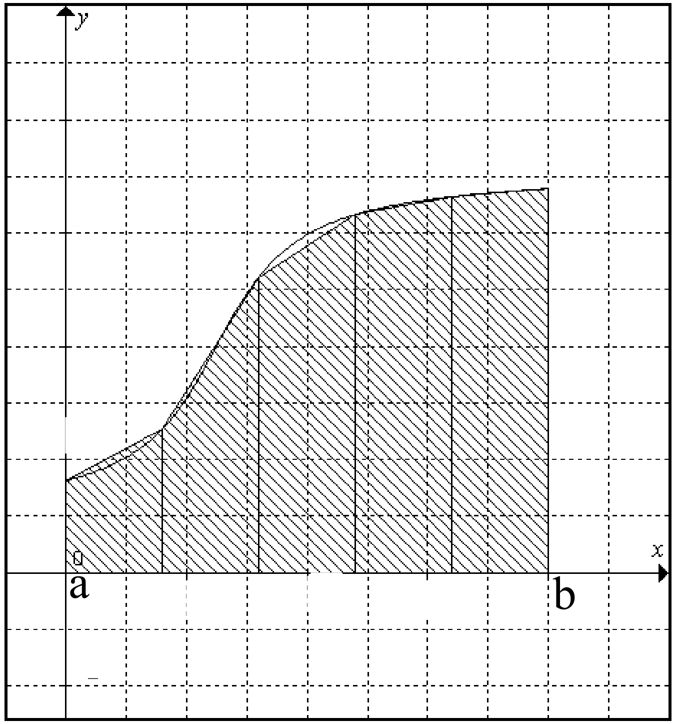
π(1+√(2))/4
        1.896118898
π/16*(2*.3827+2*
.7071+2*.9239+2*
1+2*.9239+2*.707
1+2*.3827)
        1.974255363
```



Recall that the area of a trapezoid is $A=\frac{1}{2}h(b_1+b_2)$

Height of each trapezoid: $h=\frac{b-a}{n}$

Lengths of bases of ith trapezoid: $f(x_i)$ and $f(x_{i-1})$ where $x_0=a$ and $x_n=b$.

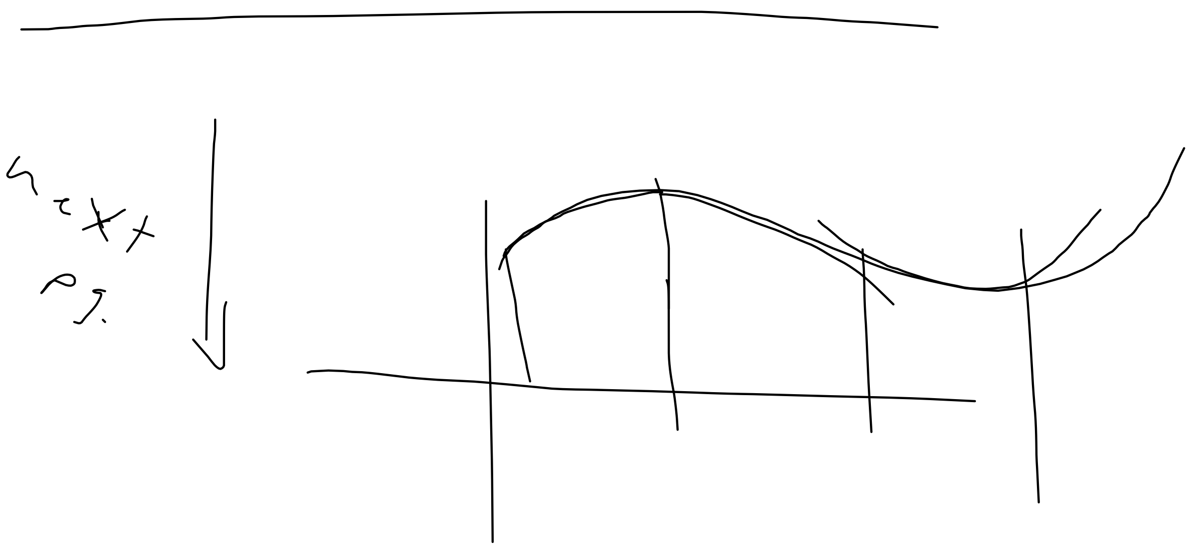
So, $\int_a^b f(x)dx$

$$=\frac{1}{2}\bullet\frac{b-a}{n}\left[[f(x_0)+f(x_1)]+[f(x_1)+f(x_2)]+[f(x_2)+f(x_3)]+[f(x_3)+f(x_4)]+[f(x_4)+f(x_5)]\right]$$

The Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_{n-1})+f(x_n)]$$

where $a=x_0$ and $b=x_n$



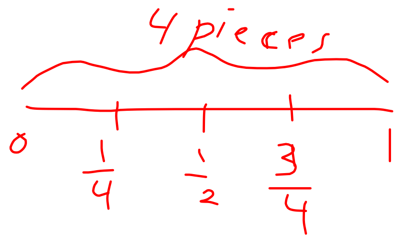
Thm. Simpson's Rule (n must be even)

$$\int_a^b f(x)dx$$

$$=\frac{b-a}{3n}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\ldots+4f(x_{n-1})+f(x_n)]$$

ex. Use Simpson's Rule to approximate $\int_0^1 x^3\,dx$ with n=4.

$$\int_0^1 x^3\,dx \approx \frac{1-0}{3\cdot 4}\left[0^3 + 4\cdot\left(\tfrac{1}{4}\right)^3 + 2\cdot\left(\tfrac{1}{2}\right)^3 + 4\left(\tfrac{3}{4}\right)^3 + 1^3\right]$$



$$= \frac{1}{12}\left[0 + \frac{1}{16} + \frac{1}{4} + \frac{27}{16} + 1\right]$$

$$= .25$$

$$\int_0^1 x^3\,dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4} - \frac{0}{4} = \frac{1}{4}$$

exact this time

Error:

1. The error in approximating $\int_a^b f(x)dx$

using the Trapezoidal Rule is: $E \leq \frac{(b-a)^3}{12n^2}[\max | f''(x) |]$

for x in [a,b].

2. The error in approximating $\int_a^b f(x)dx$

using Simpson's Rule is: $E \leq \frac{(b-a)^5}{180n^4}[\max | f^{(4)}(x) |]$

for x in [a,b].

ex. For $\int_1^2 \frac{1}{x}\,dx$, find the max error estimate for the Trapezoidal and Simpson's Rules using n=4.

$\frac{1}{x} \to x^{-1}$

$E \le \frac{(2-1)^3}{12 \cdot 4^2}\left[\max|2|\right]$

x	f''
1	2
2	1/4

$E \le \frac{1}{12 \cdot 16} \cdot 2$

$E \le .0104$

$f = x^{-1}$

$f' = -x^{-2}$

$f'' = 2x^{-3}$

$f''' = -6x^{-4}$

$0 = -6x^{-4}$

$0 \ne \frac{-6}{x^4}$

but undef'd at $x=0$ ← outside $[1,2]$

$E \le \frac{(2-1)^5}{180 \cdot 4^4}\left[\max|24|\right]$

$f^{(4)} = 24x^{-5}$

$f^{(5)} = -120x^{-6}$

x	$f^{(4)}$
1	24
2	24/32 = 3/4

$E \le .0005208$