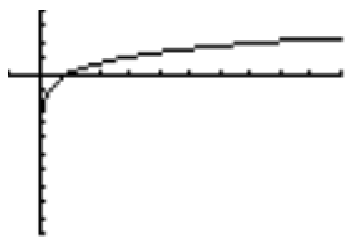
5.1 The Natural Log Function and Differentiation

Defn. $\int_1^x \frac{1}{t}\,dt = \ln x$ given that x>0.

Thm. Properties of ln

1. Domain: $(0,\infty)$ Range: $(-\infty,\infty)$
2. Increasing function
3. Continuous function
4. 1-1 function (passes the horizontal line test)
5. Concave down



6. $\ln 1 = 0$
7. $\ln\ (ab) = \ln a + \ln b$
8. $\ln \left(\frac{a}{b}\right) = \ln a - \ln b$
9. $\ln\ a^b = b \ln a$
10. $\ln x = y \ \textit{iff}\ e^y = x$

ex. Write an equivalent expression. $\ln\sqrt{3x+9}$

$= \ln(3x+9)^{\frac{1}{2}}$ $= \frac{1}{2}\ln(3x+9)$

$= \frac{1}{2}\ln 3(x+3)$

$= \frac{1}{2}[\ln 3 + \ln(x+3)]$

ex. Write an equivalent expression. $2\ln x - 3\ln y + \frac{1}{2}\ln z$

$= \ln x^2 - \ln y^3 + \ln z^{\frac{1}{2}}$

$= \ln \frac{x^2\sqrt{z}}{y^3}$

ex. Graph $\boxed{y=\ln x}+1$

Give the domain.

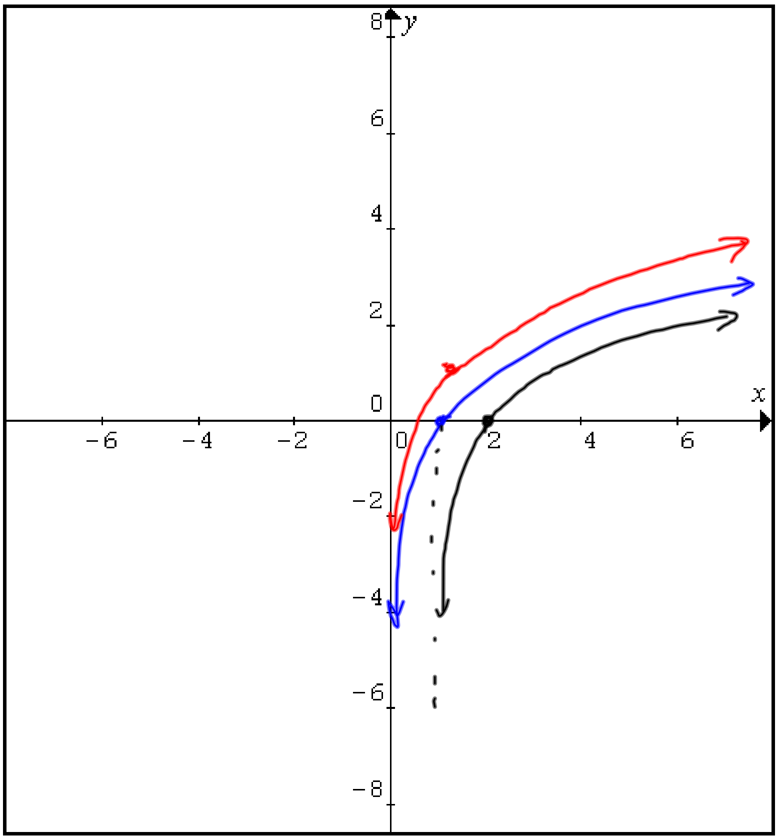
$(0,\infty)$

right 1

ex. Graph $y=\ln(x-1)$

Give the domain.

$(1,\infty)$

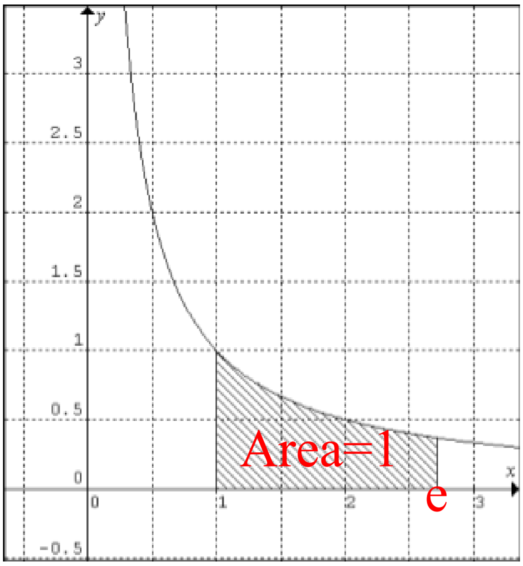


ex. Calculator active problem:
If f '(x)=sin(ln x), how many relative extrema does f have in the interval (0,2]?

$f'(x)$ changes sign twice in $(0,2]$
so there are 2 extrema for f.

Defn. The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\, dt = 1$$



e ≈ 2.71828182846

A bit of trivia:
e is called the Euler number (pronounced "OY-lur") for the Swiss mathematician Leonhard Euler.

Thm. Let u be a differentiable function of x, then

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}$

2. $\frac{d}{dx}[\ln u] = \frac{1}{u} \bullet \frac{du}{dx} = \frac{u'}{u}$

ex. If y=ln(cos(2x)), then y'= $\frac{1}{\cos(2x)} \cdot -\sin(2x) \cdot 2$

$= \frac{-2\sin(2x)}{\cos(2x)}$

$= -2\tan(2x)$

ex. The total energy expenditure per day (in excess of growth) for a species of fawn is given by

$$E(x) = 0.774 + 0.727\ln(x)$$

where x is the fawn's mass in grams and E(x) is the energy expenditure in kJ/day. Find and interpret E'(10,000), including units.

$E'(x) = .727 \cdot \frac{1}{x} = \frac{.727}{x}$

$E'(10000) = \frac{.727}{10000} = .0000727$

$\frac{kJ/day}{gram}$

The 10000g fawn's energy use increases at .0000727 kJ per day per gram.

ex. Write an equation of the line tangent to the graph of

$y = 2x^2 + \ln(3x - 8)$ at the point where x=3.

$y = 2(3)^2 + \ln(9-8) = 18 + \ln 1 = 18 \quad (3, 18)$

$y' = 4x + \frac{3}{3x-8}$

$y'(3) = 4(3) + \frac{3}{9-8} = 12 + 3 = 15$

$y - 18 = 15(x - 3) \qquad 15 = \frac{y-18}{x-3}$

ex. If tan x= ln(xy), find dy/dx.

$xy \cdot \sec^2 x = \frac{1}{\cancel{xy}} \cdot \left(1y + x\frac{dy}{dx}\right) \cdot \cancel{xy}$

$xy \sec^2 x = y + x\frac{dy}{dx}$

$xy \sec^2 x - y = x\frac{dy}{dx}$

$\frac{xy\sec^2 x - y}{x} = \frac{dy}{dx}$

ex. Find when a particle is at rest if its position on the x axis at time t is $x(t) = t\ln t$

$x'(t) = 1 \cdot \ln t + t \cdot \frac{1}{t}$

$0 = \ln t + 1$

$-1 = \ln t$

$e^{-1} = \cancel{e}^{\cancel{\ln} t}$

$e^{-1} = \cancel{e}^{\cancel{\ln} t}$

$e^{-1} = t$

$t = \frac{1}{e}$

$-1 = \ln t$

$e^{-1} = t$

ex. Find x coordinates of any points of inflection of

$$f(x)=\ln\left(x^2+2\right)$$

$$f'(x)=\frac{2x}{x^2+2}$$

$$f''(x)=\frac{2(x^2+2)-2x(2x)}{(x^2+2)^2}=\frac{-2x^2+4}{(x^2+2)^2}$$

$$0=-2x^2+4$$

$$2x^2=4$$

$$x^2=2$$

$$x=\pm\sqrt{2}$$

on $(-\infty,-\sqrt{2})$ $f''<0$

on $(-\sqrt{2},\sqrt{2})$ $f''>0$

on $(\sqrt{2},\infty)$ $f''<0$

ex. Though not tested on the AP exam, the process called logarithmic differentiation can be helpful:

$$y=\frac{(x-2)^2}{\sqrt{x^2+1}} \rightarrow \ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}}$$

$$\ln y = \ln (x-2)^2 - \ln (x^2+1)^{\frac{1}{2}}$$

$$\ln y = 2\ln (x-2) - \frac{1}{2}\ln (x^2+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2\cdot 1}{x-2} - \frac{1(\cancel{2}x)}{\cancel{2}(x^2+1)}$$

$$\frac{dy}{dx} = \left(\frac{2}{x-2} - \frac{x}{x^2+1}\right)\cdot y$$

I'll need this later, after I extend the page, so I put it here again.

$$y=\frac{(x-2)^2}{\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \left(\frac{2}{x-2} - \frac{x}{x^2+1}\right)\cdot \frac{(x-2)^2}{\sqrt{x^2+1}}$$