5.2 The Natural Logarithm Function and Integration

Thm. Let u be a differentiable function of x then,

1. $\int \frac{1}{x} dx = \ln|x| + C$
2. $\int \frac{1}{u} du = \ln|u| + C$

ex. Find the average value of the function $y = \dfrac{4}{2x+1}$ on the interval [0,2].

$$\frac{1}{2}\int_0^2 \frac{\cancel{4}\,2\cdot 2}{2x+1}\,dx = \frac{\cancel{2}}{\cancel{2}}\int_{x=0}^{x=2} \frac{2\,dx}{2x+1} = \int_1^5 \frac{du}{u} = \ln|u|\Big|_1^5$$

$u = 2x+1$

$du = 2\,dx$

$$= \ln 5 - \ln 1 = \ln 5 - 0 = \ln 5$$

ex. $\frac{1}{2}\int \frac{2x}{x^2+4}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

u

$du = 2x\,dx$

$$= \frac{1}{2}\ln|x^2+4| + C$$

$$= \ln(x^2+4)^{\frac{1}{2}} + C$$

$$= \ln\sqrt{x^2+4} + C$$

ex. Find the area under the curve y= $\dfrac{x^2+\frac{1}{3}}{x^3+x} \cdot \dfrac{3}{3} = \dfrac{3x^2+1}{3x^3+3x}$

from x=1 to x=3 and above the x axis.

$$\frac{1}{3}\int_1^3 \frac{(3x^2+1)\cdot 3}{3x^3+3x}dx = \frac{1}{3}\int_6^{90}\frac{du}{u} = \frac{1}{3}\ln|u|\Big|_6^{90}$$

$u=3x^3+3x$

$du=(9x^2+3)dx$

$$= \frac{1}{3}(\ln 90 - \ln 6)$$

$$= \frac{1}{3}\left(\ln \frac{90}{6}\right) = \frac{1}{3}\ln 15$$

or $\ln 15^{\frac{1}{3}}$ or $\ln \sqrt[3]{15}$

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(1/3)ln(90)-(1/3
)ln(6)
         .9026834004
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$$\frac{1}{3}\int \frac{3\left(x^2+\frac{1}{3}\right)}{x^3+x} \leftarrow u$$

$$du = 3x^2 + 1$$

$$\frac{1}{3}\int_2^{30} \frac{du}{u} = \frac{1}{3}\ln|u|\Big|_2^{30}$$

$$= \frac{1}{3}(\ln 30 - \ln 2)$$

ex. Can you figure out this joke? $\int \frac{d(cabin)}{cabin}dx$

ln(cabin) "natural log cabin" $+ C$

Answer

$+$ sea

ex. $\int \frac{x^2+x+1}{x^2+1}dx$

$$x^2+1 \overline{\smash{\big)}\, x^2+x+1} \quad 1 + \frac{x}{x^2+1}$$
$$-x^2 \quad -+1$$
$$x$$

$= \int 1\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}dx$

$u = x^2+1$
$du = 2x\,dx$

$= x + C_1 + \frac{1}{2}\int \frac{du}{u}$

$\quad " \quad + \frac{1}{2}\ln|u| + C_2$

$= x + C_1 + \frac{1}{2}\ln|x^2+1| + C_2$

$= x + \frac{1}{2}\ln|x^2+1| + C$

ex. $\int \frac{1(\ln x)^5}{x} dx = \int u^5 \, du = \frac{u^6}{6} + C$

$u = \ln x$

$du = \frac{1}{x} dx$

$= \frac{(\ln x)^6}{6} + C$

ex. $\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C$

$u = \cos x \quad du = -\sin x \, dx$

$= -\ln|\cos x| + C$

Integrals of Trig Functions

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = -\ln\left|\cos x\right| + C$$

$$\int \cot x \, dx = \ln\left|\sin x\right| + C$$

$$\int \sec x \, dx = \ln\left|\sec x + \tan x\right| + C$$

$$\int \csc x \, dx = -\ln\left|\csc x + \cot x\right| + C$$

ex. $\int \sqrt{1+\tan^2 x}\, dx = \int \sqrt{\sec^2 x}\, dx = \int \sec x$

$= \ln \left| \sec x + \tan x \right| + C$

ex. $\frac{1}{3}\int \frac{1 \cdot 3}{2x \ln x^3}\, dx = \frac{1}{6} \int \frac{3}{x \ln x^3}\, dx = \frac{1}{6} \int \frac{du}{u}$

$u = \ln x^3$

$du = \frac{1}{x^3} \cdot 3x^2\, dx = \frac{3}{x}\, dx$

$= \frac{1}{6} \ln |u| + C$

$= \frac{1}{6} \ln \left| \ln x^3 \right| + C$

ex. $\int \frac{\sec^2 x}{\tan x} dx$ $\leftarrow du$ $= \int \frac{du}{u} = \ln|u| + C$

u

$= \ln|\tan x| + C$

ex. $\int \frac{x^2 - 4}{x} dx = \int \left(\frac{x^2}{x} - \frac{4}{x} \right) dx = \int \left(x - 4 \cdot \frac{1}{x} \right) dx$

$= \frac{x^2}{2} - 4\ln|x| + C$