

### 5.3 Inverse functions

Defn. Functions  $g$  and  $f$  are inverses if  $f(g(x))=x$  for all  $x$  in the domain of  $g$  and  $g(f(x))=x$  for all  $x$  in the domain of  $f$ . The function  $g$  is denoted  $f^{-1}$ .

ex. Verify that  $f(x)=4x^3+1$  and  $g(x)=\sqrt[3]{\frac{x-1}{4}}$  are inverses.

Thm. The graph of  $f$  contains the point  $(a,b)$  iff the graph of  $f^{-1}$  contains the point  $(b,a)$ .

Thm.

1. A function has an inverse iff it is 1-1 (passes the horizontal line test).
2. If  $f$  is strictly monotonic (incr. or decr.) on its entire domain, then it's 1-1 and therefore has an inverse.

(Review) Finding an inverse:

1. Switch  $x$  and  $y$  ( $f(x)$ ).
2. Solve for  $y$ , which is then  $f^{-1}$
3. Define the domain of  $f^{-1}$  as the range of  $f$ .
4. Verify that  $f$  and  $f^{-1}$  are inverses.

ex. Find the inverse of  $f(x) = \sqrt[4]{3-x}$

Teacher note: Explore inverses on the APCD.

Thm. Let  $f$  be a function whose domain is  $I$ . If  $f$  has an inverse, then

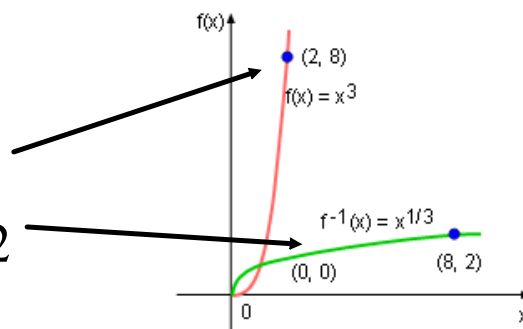
1.  $f$  is cont's on  $I \rightarrow f^{-1}$  is cont's on its domain.
2.  $f$  is incr'g (decr'g) on  $I \rightarrow f^{-1}$  is incr'g (decr'g) on its domain.
3.  $f$  is diff'ble at  $c$  and  $f'(c)$  is not 0  $\rightarrow f^{-1}$  is diff'ble at  $f(c)$ .

Thm. The derivative of an inverse function

Let  $f$  be a function that's diff'ble on  $I$  and let  $g$  be the inverse of  $f$ . Then  $g$  is diff'ble at any  $x$  for which  $f'(g(x)) \neq 0$  and  $g'(x) = 1/f'(g(x))$ .

Graphically:

slope at of  $f$  at  $(2,8)$  is 12  
 slope of  $f^{-1}$  at  $(8,2)$  is  $1/12$



Numerically:  $f'(a)=m$  at  $(a,b)$  and  $g=f^{-1}$ ,  
 then  $g'(b)=1/m$  at  $(b,a)$ .

Analytically: Write  $f(g(x))=x$  with " $g$ " instead of " $g(x)$ " for simplicity. Perform implicit differentiation and solve for  $g'(x)$ .

Written: The derivative of an inverse function at a point is the reciprocal of the derivative of original function at its corresponding point.

ex. If  $f(1)=-3$ ,  $f'(1)=4$ , and  $g$  is the inverse of  $f$ , then what is  $g'(-3)$ ?

ex. Suppose  $f(3)=5$ ,  $f'(3)=4$ ,  $f(5)=7$ ,  $f'(5)=2$ , and  $g$  is the inverse of  $f$ . What is  $g'(5)$ ?

ex. Let  $f(x)=x^3+x$ . If  $g(x)=f^{-1}(x)$  and  $g(2)=1$ ,  
what is  $g'(2)$ ?

ex. If  $g(x)=2x+5$ , then  $\frac{d}{dx}(g^{-1}(x)) =$

ex. Use the calculator to help you find  $g'(2)$  where  $g$  is the inverse of  $f(x) = x^5 - x^3 + 2x$

ex. Find the slope of  $f^{-1}$  at  $x=3$  if  $f(x) = \sqrt{2x+5}$



ex. Find the derivative of the inverse of

$$f(x) = x^5 + 7x^2$$