5.3 Inverse functions

Defn. Functions g and f are <u>inverses</u> if f(g(x))=x for all x in the domain of g and g(f(x))=x for all x in the domain of f. The function g is denoted f^{-1}.

ex. Verify that f(x)= $4x^3+1$ and g(x)= $\sqrt[3]{\frac{x-1}{4}}$ are inverses.

$$f(g(x)) = 4\left(\sqrt[3]{\frac{x-1}{4}}\right)^3 + 1$$

$$= \cancel{4}\left(\frac{x-1}{\cancel{4}}\right) + 1$$

$$= x - 1 + 1$$

$$= x$$

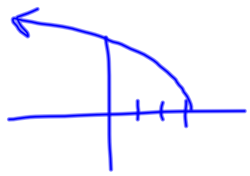
$$g(f(x)) = \sqrt[3]{\frac{\cancel{4}x^3 \cancel{+1} \cancel{-1}}{\cancel{4}}}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

Thm. The graph of f contains the point (a,b) iff the graph of f^{-1} contains the point (b,a).

Thm.
1. A function has an inverse iff it is 1-1 (passes the horizontal line test).
2. If f is strictly monotonic (incr. or decr.) on its entire domain, then it's 1-1 and therefore has an inverse.



(Review) Finding an inverse:

1. Switch x and y (f(x)).
2. Solve for y, which is then f^{-1}
3. Define the domain of f^{-1} as the range of f.
4. Verify that f and f^{-1} are inverses.

$3 - x \geq 0$

$-x \geq -3$

$x \leq 3$

D: $(-\infty, 3]$

R: $[0, \infty)$

ex. Find the inverse of f(x)= $\sqrt[4]{3-x}$

$y = \sqrt[4]{3-x}$

$x = \sqrt[4]{3-y} \rightarrow x^4 = 3 - y \longrightarrow y = 3 - x^4$ for $x \in [0, \infty)$

Range: $(-\infty, 3]$

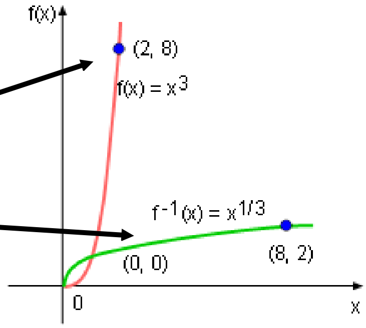
Teacher note: Explore inverses on the APCD.

Thm. Let f be a function whose domain is I. If f has an inverse, then

1. f is cont's on I → f^{-1} is cont's on its domain.
2. f is incr'g (decr'g) on I → f^{-1} is incr'g (decr'g) on its domain.
3. f is diff'ble at c and f '(c) is not 0 → f^{-1} is diff'ble at f(c).

Thm. The derivative of an inverse function
Let f be a function that's diff'ble on I and let g be the inverse of f. Then g is diff'ble at any x for which $f'(g(x))\neq 0$ and $g'(x) = 1/f'(g(x))$.

Graphically:
slope at of f at (2,8) is 12
slope of f^{-1} at (8,2) is 1/12



Numerically: $f'(a)=m$ at (a,b) and $g=f^{-1}$, then $g'(b)=1/m$ at (b,a).

Analytically: Write $f(g(x))=x$ with "g" instead of "g(x)" for simplicity. Perform implicit differentiation and solve for $g'(x)$.

Written: The derivative of an inverse function at a point is the reciprocal of the derivative of original function at its corresponding point.

ex. If f(1)=-3, f '(1)=4, and g is the inverse of f, then what is g'(-3)? $= \frac{1}{4}$

$f(\ ) = -3$

$f(1) = -3$

$f'(1) = 4$

ex. Suppose f(3)=5, f '(3)=4, f(5)=7, f '(5)=2, and g is the inverse of f. What is g'(5)? $= \frac{1}{4}$

$f(\ ) = 5$

$f(3) = 5 \rightarrow f'(3) = 4$

ex. Let $f(x)=x^3+x$. If $g(x)=f^{-1}(x)$ and $g(2)=1$, what is $g'(2)$?

$f'(x)=3x^2+1$

$f(1)=2$

$f'(1)=?$

$f'(1)=3\cdot 1^2+1=4$

$g'(2)=\frac{1}{4}$

ex. If $g(x)=2x+5$, then $\frac{d}{dx}\left(g^{-1}(x)\right)=$

$g'=f$

$f(g)=x$

$f'(g)\cdot g'=1$

$g'=\frac{1}{f'(g)}$

$f(2x+5)=x$

$f'(2x+5)\cdot 2=1$

$f'(2x+5)=\frac{1}{2}$

ex. Use the calculator to help you find g'(2) where g is the inverse of $f(x) = x^5 - x^3 + 2x$

$f(1) = 2$

so $g(2) = 1$

$f'(x) = 5x^4 - 3x^2 + 2$

$f'(1) = 5 \cdot 1^4 - 3 \cdot 1^2 + 2$

$= 4$

so $g'(2) = \frac{1}{4}$

ex. Find the slope of $\boxed{f^{-1} \text{ at } x=3}$ if $f(x) = \sqrt{2x+5}$

$f(2) = 3$ $f'(2) = \frac{1}{3}$

$f^{-1}(3) = 2$

slope of f^{-1} at $x = 3$ is 3

ex. Find the derivative of the inverse of $f(x) = x^5 + 7x^2$

Let g and f be inverses and let

$y = f(x) = x^5 + 7x^2 \longrightarrow f(x) = y$ means f contains (x, y)

then $\frac{dy}{dx} = 5x^4 + 14x \longrightarrow \frac{dy}{dx}$ is $f'(x)$ or y'

so $\frac{dx}{dy} = \frac{1}{5x^4 + 14x}$

which can be written

$g'(x) = \frac{1}{5x^4 + 14x}$

$\longrightarrow \frac{dx}{dy}$ is the derivative of g

OK, that's our answer