5.5 Bases other than e and applications

Recall your exponent rules:

For a positive real number a, a≠1, and any real x:

1. $a^0=1$
2. $a^x a^y = a^{x+y}$
3. $a^x / a^y = a^{x-y}$
4. $(a^x)^y = a^{xy}$

Definition of $\log_a$ (Log base a)

$$\log_a x = \frac{\ln x}{\ln a}$$

Properties:

1. $\log_a 1 = 0$
2. $\log_a xy = \log_a x + \log_a y$
3. $\log_a x/y = \log_a x - \log_a y$
4. $\log_a x^y = y \log_a x$

Properties of log and its inverse

1. $\log_a x = y$ iff $a^y = x$

2. $\cancel{\log_a a}^x = x$

3. $\cancel{a^{\log_a}} x = x$

ex. Solve $3^x = \dfrac{1}{81}$

$3^x = \dfrac{1}{3^4}$

$3^x = 3^{-4}$

$x = -4$

$\ln 3^x = \ln \frac{1}{81}$

$x \ln 3 = \ln \frac{1}{81}$

$x = \dfrac{\ln \frac{1}{81}}{\ln 3} = \dfrac{\overset{0}{\cancel{\ln 1}} - \ln 81}{\ln 3} = \dfrac{-\ln 81}{\ln 3}$

ex. Solve $\log_2 x = -4$

$2^{-4} = x$

$\frac{1}{16} = x$

ex. Solve $\log_2 64 = 3x+5$

$\log_2 2^6 = 3x+5$

$6 = 3x+5$

$1 = 3x$

$\frac{1}{3} = x$

$2^{3x+5} = 64$

$2^{3x+5} = 2^6$

Derivatives

1. $\frac{d}{dx}\left[a^x\right]=\ln a \cdot a^x$

here's how:

$$\boxed{\begin{aligned} y &= a^x \\ \ln y &= x\ln a \\ \frac{y'}{y} &= 1 \bullet \ln a \\ y' &= y \bullet \ln a \\ y' &= a^x \bullet \ln a \end{aligned}}$$

2. $\frac{d}{dx}\left[a^u\right] = \ln a \bullet a^u \bullet u'$

3. $\frac{d}{dx}\left[\log_a x\right] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{x\ln a}$

4. $\frac{d}{dx}\left[\log_a u\right] = \frac{u'}{u\ln a}$

Integral formula

$$\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$$

ex. If $y = 2^{3x^2}$, then $y' = 2^{3x^2} \cdot \ln 2 \cdot 6x$

$$\ln y = \ln 2^{3x^2}$$

$$\ln y = 3x^2 \cdot \ln 2$$

$$\frac{y'}{y} = 6x \cdot \ln 2 \longrightarrow y' = 6x \cdot \ln 2 \cdot 2^{3x^2}$$

ex. If $f(x) = \log_{10} \frac{x^2}{x-1}$, then $f'(x) = \frac{1}{\ln 10 \, \frac{x^2}{x-1}} \cdot \frac{2x(x-1) - x^2(1)}{(x-1)^2}$

$$= \frac{\ln \frac{x^2}{x-1}}{\ln 10}$$

$$f'(x) = \frac{1}{\ln 10 \left(\frac{x^2}{x-1}\right)} \cdot \frac{2x(x-1) - x^2(1)}{(x-1)^2}$$

$$= \frac{\cancel{x-1}}{x^{\cancel{2}} \cdot \ln 10} \cdot \frac{x^{\cancel{2}} - 2\cancel{x}}{(x-1)^{\cancel{2}}} = \frac{x-2}{x \ln 10 (x-1)}$$

ex. $\frac{1}{6}\int 6x7^{3x^2}\,dx = \frac{1}{6}\int 7^u\,du = \frac{1}{6}\cdot\frac{1}{\ln 7}\cdot 7^u + C$

$u = 3x^2$

$du = 6x\,dx$

$= \frac{7^{3x^2}}{6\ln 7} + C$