5.7 Inverse Trig functions and Integration

Thm. Let u be a diff'ble function of x and let a be a positive constant.

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right)+C$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{u}{a}\right)+C$$

If any are negative, it is the cofunction.

ex. $\int \frac{dx}{\sqrt{4-x^2}}$ $= \arcsin\left(\frac{x}{2}\right) + C$

a^2 u^2

$a^2 = 4$ $u^2 = x^2$

$a = 2$ $u = x$

$du = dx$

ex. $\int \frac{\cos x}{2+\sin^2 x}\,dx = \int \frac{du}{a^2+u^2}$

a^2 u^2

$= \frac{1}{\sqrt{2}} \arctan\left(\frac{\sin x}{\sqrt{2}}\right) + C$

$a = \sqrt{2}$ $u = \sin x$

$du = \cos x \, dx$

ex. $\displaystyle\int \frac{2\,dx}{2x\sqrt{4x^2-9}} = \frac{1}{3}\,\mathrm{arcsec}\left(\frac{2x}{3}\right) + C$

u^2 $\quad a^2$

$u = 2x \qquad a = 3$

$du = 2\,dx$

ex. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{e^x\,dx}{e^x\sqrt{e^{2x}-1}} = \frac{1}{1}\,\mathrm{arcsec}\left(\frac{e^x}{1}\right) + C$

u^2 $\quad a^2$

$u = e^x \qquad a = 1$

$du = e^x\,dx$

$= \mathrm{arcsec}(e^x) + C$

ex. $\int \frac{dx}{x^2-4x+7} = \int \frac{dx}{x^2-4x+4+7-4}$

$+4 \quad -4$

$\int \frac{dx}{(x-2)^2+3} = \frac{1}{\sqrt{3}} \arctan\left(\frac{x-2}{\sqrt{3}}\right) + C$

u^2 $\qquad a^2$

$a = \sqrt{3}$

$u = x-2$

$du = dx$

ex. $\int \frac{(x+2)}{\sqrt{4-x^2}}dx = -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{4-x^2}} + \int \frac{2\,dx}{\sqrt{4-x^2}}$

$\sqrt{4-x^2}$: $4 = a^2$, $x^2 = u^2$

$a = 2$
$u = x$
$du = dx$

$u = 4 - x^2$
$du = -2x\,dx$

$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} \qquad + 2\arcsin\left(\frac{x}{2}\right) + C_2$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= -\frac{1}{2}\cdot 2u^{\frac{1}{2}} + C_1$

$= -\sqrt{4-x^2} + C_1 + 2\arcsin\left(\frac{x}{2}\right) + C_2$

$= -\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right) + C$

ex. $\int \frac{1}{\sqrt{3x-x^2}}\,dx =$ $\int \frac{1}{\sqrt{\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$

$$-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$\frac{1}{2}$ $\square^2$

$$-\left(\left(x-\frac{3}{2}\right)^2-\frac{9}{4}\right)$$

$$\left(\frac{9}{4}-\left(x-\frac{3}{2}\right)^2\right)$$

a^2 u^2

$a=\frac{3}{2}$ $u = x-\frac{3}{2}$

$du = dx$

$$= \arcsin\left(\frac{x-\frac{3}{2}}{\frac{3}{2}}\right)+C$$

$$= \arcsin\left(\frac{2x-3}{3}\right)+C$$

ex. $\displaystyle\int \frac{2x+7}{x^2+2x+5}\,dx = \int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{5\,dx}{x^2+2x+5}$

$u = x^2+2x+5$

$du = 2x+2\ dx$

$= \displaystyle\int \frac{du}{u} + 5\int \frac{dx}{\underbrace{x^2+2x+1}+\underbrace{5-1}}$

$= \ln|u|$

$= \ln|x^2+2x+5| \quad + 5\displaystyle\int \frac{dx}{\underbrace{(x+1)^2}_{v^2}+\underbrace{4}_{a^2}}$

$v = x+1 \qquad a = 2$

$dv = dx$

$+\ln|x^2+2x+5| + 5\cdot\frac{1}{2}\arctan\left(\frac{x+1}{2}\right)+C$

ex. $\int \frac{x-3}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx - 3\int \frac{dx}{x^2+1}$

u^2 a^2

$u = x^2+1$

$du = 2x\,dx$

$u = x$ $a = 1$

$du = dx$

$\frac{1}{2}\int \frac{du}{u}$ $-3\left(\frac{1}{1}\right)\arctan\left(\frac{x}{1}\right)$

$\frac{1}{2}\ln|u|$

$\frac{1}{2}\ln|x^2+1| - 3\arctan x + C$