6.2 Differential Equations: Growth and Decay

Thm. $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x} = e$

Continuous Compounding Formula

amount after some time

interest rate as a decimal

$A=Pe^{rt}$

time (years)

principal invested

ex. Use the Continuous Compounding Formula to determine the time needed to double an investment at 6% interest.

$$2 = 1e^{.06t}$$

$$\ln 2 = \cancel{\ln e}^{.06t}$$

$$\ln 2 = .06t$$

$$\frac{\ln 2}{.06} = t = 11.552 \text{ years}$$

Time to double is often estimated with the “Rule of 72”

$$2=1e^{rt}$$
$$\ln 2=\ln e^{rt}$$
$$\ln 2=rt$$
$$.6931471806=rt$$

If r is a percent, rather than a decimal,
69.31471806=rt

So the rule is actually the “Rule of 69.31471806”
but we use 72 because it has more factors.

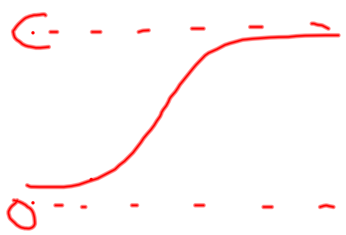
GNAW on Differential Equations

Written/Verbally: "rate of change"

DE	**verbiage**	**setting**
$dy/dt = 0$	growth rate is 0	y is constant
$dy/dt = ky$	growth of y is proportional to y, with proportional constant k	continuously compounded interest
$dy/dt = kt$	growth rate of y at time t is proportional to t	y is the height of a free-falling object
$y'' = k$	y has constant acceleration	y is the height of a free-falling object
$dy/dt = k(y-A)$	y changes at a rate proportional to the difference between y and A	y is the temperature of an object in a room at A degrees, the ambient temperature, and k is a negative constant for a cooling object or a positive constant for a warming object. This is called Newton's Law of Cooling.
$dy/dt = -ky^{1/2}$	dy/dt = y decreases at a rate proportional to the square root of y	y is the depth of liquid in a tank being drained out of a valve at the bottom of the tank. This is called Torricelli's Law.

BC only:

$dy/dt = ky(C-y)$ or $dy/dt = (y/k)(1-y/C)$	the rate of change of y is proportional both to the value of y itself and to the difference between the carrying capacity, C, and the value of y	y is the number of people who are infected with a contagious disease at time t



ex. By U.S. law, yogurt must contain 100 million bacteria per gram. At noon, some sterilized milk is inoculated with a yogurt culture so that the milk contains 400 bacteria per gram. Suppose the bacteria growth rate is proportional to the number of bacteria present and that at 1pm, there are 1600 bacteria per gram. At 7pm, how many bacteria are there per gram? At what time does the culture legally become yogurt?

$t=0$

y

$$\frac{dy}{y\,dt} = \frac{ky}{y}$$

$$\int \frac{dy}{y} = \int k\,dt = k\int 1\,dt$$

$$\ln y = kt + C$$

$$\ln 400 = k\cdot 0 + C$$

$$\ln 400 = C$$

$$\ln 1600 = k\cdot 1 + \ln 400$$

$$\ln 1600 - \ln 400 = k$$

$$\cancel{\ln}\, y = k\cdot 7 + C$$

$$\cancel{e} \quad e$$

$$y = e^{7k+C} = 6{,}553{,}600$$

$$\ln 100{,}000{,}000 = kt + C$$

$$\frac{\ln 100{,}000{,}000 - C}{k} = t$$

ex. A student is preparing a batch of yogurt for a lab and the bacteria growth rate is proportional to the number of bacteria present. Because of time constraints, the student needs the batch ready in 6 hours. If, six hours after beginning, the bacterial load is to be 100 million bacteria per gram and the load was 50 million bacteria 5 and a half hours after inoculation, what bacterial load would the student need to begin with?

$$\frac{dy}{dt} = ky \rightarrow \ln y = kt + C \rightarrow y = e^{kt+C}$$

$$\ln 100{,}000{,}000 = 6k + C \rightarrow \ln 100\text{mil.} = 6k + C$$

$$\ln 50{,}000{,}000 = 5.5k + C \qquad \ln 100\text{mil.} - 6k = C$$

$$\ln 100\text{mil.} - \ln 50\text{mil.} = .5k$$

$$1.386 = k \qquad 10.103 = C$$

$$\ln y = kt + C$$

$$\ln y = k \cdot 0 + C$$

$$\ln y = C$$

$$y = e^C = e^{10.103}$$

$$y = 24{,}414.0625$$

bacteria / gram

A 250°F meatball is removed from a simmering pan and set to cool on a plate in a 70°F = A room. In 3 minutes, it is 200°F. How long before it falls below 140°F and into the "danger zone" within which bacteria can thrive?

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y = temp. of meatball

$$\frac{dy}{dt} = k(y - A) \quad \text{temp. of surroundings}$$

t	y
0	250
3	200
?	140

$$dy = k(y - \cancel{A}^{70})\,dt$$

$$\int \frac{1\,dy}{y-70} = k\int dt$$

$$\ln|y-70| = kt + C$$

$$\ln(250-70) = 0 + C$$

$$\ln 180 = C$$

$$\ln(200-70) = 3k + C$$

$$\ln 130 - C = 3k$$

$$\frac{\ln 130 - C}{3} = k$$

$$\ln(140-70) = kt + C$$

$$\frac{\ln 70 - C}{k} = t$$

$t = 8.707$ minutes

The number of radioactive atoms y remaining after t days in sample of polonium-210 that starts with y_0 radioactive atoms is

$$y = y_0 e^{-0.005t}$$

$$1 \quad 2$$

a. Find the element's half life.

$$\frac{1}{2} = e^{-.005t}$$

$$\ln \frac{1}{2} = -.005t$$

$$t = \frac{\ln \frac{1}{2}}{-.005} = 138.629 \text{ days}$$

b. A sample of polonium-210 is not useful for some experiment after 95% of the radioactive nuclei have disintegrated. For about how many days will the sample be of use?

$$5 = 100e^{-.005t}$$

$$.05 = e^{-.005t}$$

$$\ln .05 = -.005t$$

$$\frac{\ln .05}{-.005} = t$$

$$t = 599.146 \text{ days}$$