6.3 Differential Equations: Separation of Variables

Separation of Variables: Put x and dx on one side, y and dy on the other side, integrate.

ex. Find a general solution for $(x^2+4)\frac{dy}{dx} = xy$

$(x^2+4)\,dy = xy\,dx$

$\int \frac{dy}{y} = \frac{1}{2}\int \frac{2x\,dx}{x^2+4} \leftarrow u$

$\ln|y| = \frac{1}{2}\ln|u| + C_1$ $\quad du = 2x\,dx$

$\ln|y| = \frac{1}{2}\ln|x^2+4| + C_1$

$e^{\ln|y|} = e^{\ln\sqrt{x^2+4} + C_1}$

$|y| = e^{\ln\sqrt{x^2+4}} \cdot e^{C_1}$

$|y| = \sqrt{x^2+4} \cdot C$

$y = \pm C\sqrt{x^2+4}$

$|x| = 4$

$x = \pm 4$

$x^4 \cdot x^2 = x^{4+2}$

GNAW on Differential Equations

Analytically: separation of variables and integration

Differential Equation (DE)

ex. solve dy/dt = -y$^{1/2}$

$$\frac{dy}{dt} = -y^{\frac{1}{2}}$$

$$\frac{dy}{y^{\frac{1}{2}}} = -1\,dt$$

$$\int y^{-\frac{1}{2}}\,dy = -1\int dt$$

$$2y^{\frac{1}{2}} = -1t + C$$

$$y^{\frac{1}{2}} = -\frac{t}{2} + \frac{C}{2}$$

$$y = \left(-\frac{t}{2} + \frac{C}{2}\right)^2$$

Initial Value Problem (IVP)

ex. solve dy/dt = -y$^{1/2}$ if t=1 when y=4

$$\sqrt{4} = \sqrt{\left(-\frac{1}{2} + \frac{C}{2}\right)^2}$$

$$2 = -\frac{1}{2} + \frac{C}{2}$$

$$2\frac{1}{2} = \frac{C}{2}$$

$$5 = C$$

so, $y = \left(-\frac{t}{2} + \frac{5}{2}\right)^2$

ex. Solve analytically: Suppose the population y of a hive of wasps is growing at a rate proportional to the population. On May 1, there were 10 wasps and on May 31 there were 50. If growth continues like this, how long after May 1 will the population reach 100 wasps?

	t	y
5/1	0	10
5/31	30	50
	?	100

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = k\int dt$$

$$\ln|y| = kt + C_1$$

$$|y| = e^{kt + C_1}$$

$$y = \pm e^{kt} \cdot \underbrace{e^{C_1}}_{C}$$

$$y = \pm C e^{kt}$$

$$10 = Ce^{k \cdot 0}$$

$$10 = C \cdot 1$$

$$50 = 10e^{k \cdot 30}$$

$$5 = e^{30k}$$

$$\ln 5 = 30k$$

$$\frac{\ln 5}{30} = k$$

$$100 = 10e^{kt}$$

$$10 = e^{kt}$$

$$\ln 10 = kt$$

$$\frac{\ln 10}{k} = t$$

$t = 42.92$ days after May 1

ex. Find the particular solution of dy/dx = x/y through (-2,-1) and identify the domain.

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\frac{(-1)^2}{2} = \frac{(-2)^2}{2} + C$$

$$\frac{1}{2} = 2 + C$$

$$-\frac{3}{2} = C$$

$$\frac{y^2}{2} = \frac{x^2}{2} - \frac{3}{2}$$

$$y^2 = x^2 - 3$$

$$y = \pm\sqrt{x^2-3}$$

$$\cancel{y = \sqrt{x^2-3}} \qquad y = -\sqrt{x^2-3}$$

$\cancel{(-2,1)}$

Domain: $x^2 - 3 \ge 0$

$$x^2 \ge 3$$

$$x \le -\sqrt{3} \text{ or } x \ge \sqrt{3}$$

$$\boxed{(-\infty, -\sqrt{3})} \cup \cancel{(\sqrt{3}, \infty)}$$

BC topic: Logistic growth (slow-fast-slow)

http://math.jccc.net:8180/webMathematica/MSP/mmartin/logisticDE

Your book uses: $\frac{dy}{dt}=ky\left(1-\frac{y}{L}\right)$

but you might see things arranged like this: $\frac{dy}{dt}=\left(\frac{k}{L}\right)y\left(L-y\right)$

The solution is: $y=\frac{L}{1+be^{(-kt)}}$

where $b=\frac{y_0-L}{-y_0}=\frac{L-y_0}{y_0}$

L is the carrying capacity
y is the number in the population
t is time
k is the growth constant

ex. A pond has a carrying capacity of 500 fish. Assume the population grows logistically with growth constant k=0.4 for time t in months.
a. Find the fish population model y(t) if the initial population is 50 fish.
b. How long does it take for the fish population to reach 250?

$$\frac{dy}{dt} = ky\left(1-\frac{y}{L}\right)$$

$$y=\frac{L}{1+be^{-kt}}$$

$$\frac{dy}{dt} = .4y\left(1-\frac{y}{500}\right)$$

a. $y(t) = \dfrac{500}{1+9e^{-.4t}}$

$$\frac{500-50}{50} \rightarrow b = \frac{L-y_0}{y_0}$$

b. $250 = \dfrac{500}{1+9e^{-.4t}}$

$$1 = \frac{2}{1+9e^{-.4t}}$$

$$1+9e^{-.4t} = 2$$

$$9e^{-.4t} = 1$$

$$e^{-.4t} = \frac{1}{9}$$

$$-.4t = \ln\frac{1}{9}$$

$$t = \frac{\ln\frac{1}{9}}{-.4} \approx 5.493 \text{ months}$$

BC ex. The growth rate of a population of wolves in a newly established preserve is modeled by dp/dt = .008p(100-p), where t is measured in years.

$$\frac{dy}{dt}=\left(\frac{k}{L}\right)y\left(L-y\right)$$

a. What is the carrying capacity for the wolves in this preserve?
100 wolves

Answer

b. What is the wolf population when the population if growing the fastest?

Answer

When it is half the carrying capacity, 50 wolves.

c. What is the rate of change of the population when it is growing the fastest?
When p=50, dp/dt = .008(50)(100-50) = 20 wolves per year. So the growth rate is about 20 wolves that year (the derivative is an instantaneous growth rate).

d. If p(0)=3, what value does p approach as t grows infinitely large?
100, since that's the carrying capacity

Answer