8.8 Improper Integrals

Defn.

1. If f is continuous on [a,∞) then

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

2. If f is continuous on (-∞,b] then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

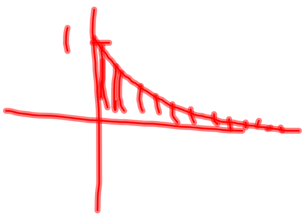
3. If f is continuous on (-∞,∞) then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

If the limit exists then the integral <u>converges</u>, otherwise, it <u>diverges</u> (tends toward infinity).

ex. $\int_0^\infty e^{-x}dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$

$$= \lim_{b\to\infty} -e^{-x}\Big|_0^b$$

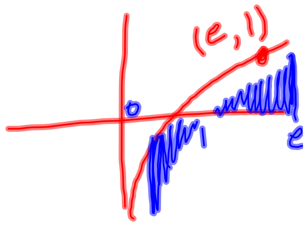


$$= \lim_{b\to\infty} -e^{-b} + +e^{0}$$

$$= \lim_{b\to\infty} -\frac{1}{e^{b}} + 1$$

$$0 + 1 = 1$$

ex. $\displaystyle\int_0^e \ln x\,dx = \lim_{a\to 0^+} \int_a^e \underset{L}{\ln x}\,\underset{A}{dx}$



$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$= \displaystyle\lim_{a\to 0^+} x\ln x \Big|_a^e - \int_a^e \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx$

$= \displaystyle\lim_{a\to 0^+} (x\ln x - x)\Big|_a^e$

$= \displaystyle\lim_{a\to 0^+} (e\ln e - e) - (a\ln a - a)$

$\qquad 0 \qquad\qquad 0(-\infty) - 0$

$\qquad 0 \quad - (0 - 0)$

$= 0$

$\displaystyle\lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} \quad \frac{-\infty}{\infty}$

$= \displaystyle\lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$

$= \displaystyle\lim_{a\to 0^+} \frac{1}{a}\cdot\frac{-a^2}{1}$

$= \displaystyle\lim_{a\to 0^+} -a$

ex. $\displaystyle\int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{2x}} = \lim_{a\to-\infty}\int_a^0 \frac{e^x\,dx}{1+e^{2x}} + \lim_{b\to\infty}\int_0^b \frac{e^x\,dx}{1+e^{2x}}$

$u = e^x$

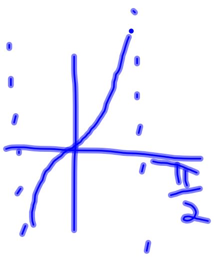
$du = e^x\,dx$

from ch. 5.6 → $\displaystyle\int \frac{du}{1+u^2} = \frac{1}{1}\arctan\frac{u}{1}$

$a^2 \quad a = 1$

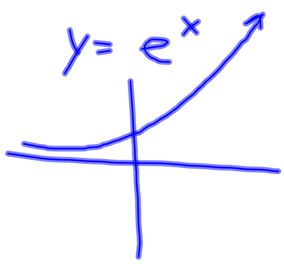
$\displaystyle\lim_{a\to-\infty} \arctan e^x \Big|_a^0 + \lim_{b\to\infty} \arctan e^x \Big|_0^b$

$\displaystyle\lim_{a\to-\infty} \arctan 1 - \arctan e^a + \lim_{b\to\infty} \arctan e^b - \arctan 1$



$\cancel{\frac{\pi}{4}} - 0 + \frac{\pi}{2} \cancel{-\frac{\pi}{4}}$

$= \frac{\pi}{2}$



ex. $\int_1^{\infty} (1-x)e^{-x}\,dx = \lim_{b\to\infty}$

u $\quad dv = e^{-x}$

$du = -1\,dx \quad v = -e^{-x}$

$(1-x)(-e^{-x}) - \int +1(+e^{-x})\,dx$

$(1-x)(-e^{-x}) + (+e^{-x})$

$-e^{-x} + xe^{-x} + e^{-x}$

$x e^{-x} \Big|_1^b$

$= \lim_{b\to\infty} \dfrac{x}{e^x}\Big|_1^b$

$= \lim_{b\to\infty} \dfrac{b}{e^b} - \dfrac{1}{e^1}$

$0 - \dfrac{1}{e}$

$-\dfrac{1}{e}$

Thm. $$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \\ \text{diverges otherwise} \end{cases}$$

ex. Find the volume of the solid formed by revolving the function y=1/x around the x axis for x≥1.

$$\pi\int_1^{\infty}\left(\frac{1}{x}-0\right)^2 dx$$

$$\lim_{b\to\infty}\pi\int_1^b \frac{1}{x^2}\,dx = \pi\cdot\frac{1}{2-1} = \pi$$

$$\lim_{x\to\infty}\frac{1}{x}=0$$

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}=\infty$$

$$\lim_{x\to\infty}\frac{1}{x^2}=0$$

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}=1$$

$$\lim_{x\to\infty}\frac{1}{x^3}=0$$

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^3}=\frac{1}{2}$$

ex. $\displaystyle\int_4^5 \frac{dx}{(x-4)^3} = \lim_{a\to 4^+} \int_a^5 \underbrace{(x-4)}_{u}{}^{-3}\,dx$

$du = 1\,dx$

$\displaystyle = \lim_{a\to 4^+} \left.\frac{(x-4)^{-2}}{-2}\right|_a^5 = \lim_{a\to 4^+} \frac{1}{-2(5-4)^2} + \frac{1}{+2(a-4)^2}$

$\displaystyle = \frac{1}{-2} + \infty = \infty$

ex. $\displaystyle\int_5^\infty \frac{dx}{(x-4)^3} = \lim_{b\to\infty} \int_5^b (x-4)^{-3}\,dx$

$\displaystyle = \lim_{b\to\infty} \left.\frac{(x-4)^{-2}}{-2}\right|_5^b$

$\displaystyle = \lim_{b\to\infty} \frac{1}{-2(b-4)^2} + \frac{1}{+2(5-4)^2}$

$\displaystyle = 0 + \frac{1}{2}$

$\displaystyle = \frac{1}{2}$