9.10 Taylor and Maclaurin Series

Thm. Short form: A power series representing a function is its Taylor series.

Thm. If f is represented by a power series $f(x) = \sum a_n (x-c)^n$

for all x in an open interval I containing c, then $a_n = \frac{f^{(n)}(c)}{n!}$

and f(x)=

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \dots$$

Defn. If a function has derivative of all orders at x=c, then that series is called the Taylor series for f at c.

If c=0, the series is called the Maclaurin series for f.

From the previous section, $\frac{3}{2-x} = \sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{x}{2}\right)^n$

Find some terms using the theorem.

$\frac{3}{2} + \frac{3x}{4} + \frac{3x^2}{8} + \frac{3x^3}{16} + \cdots$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = \frac{f(c)}{0!} + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots$$

ex. Find the Maclaurin series for f(x)= $\frac{1}{1-x}$

<u>Know</u> this and the Maclaurin series for sin, cos, and e^x.

$f(x) = (1-x)^{-1}$ $\quad f(0) = 1$ $\quad \frac{1}{0!} = 1$

$f'(x) = +(1-x)^{-2}$ $\quad f'(0) = 1$ $\quad \frac{1}{1!} = 1$

$f''(x) = +2(1-x)^{-3}$ $\quad f''(0) = 2$ $\quad \frac{2}{2!} = 1$

$f'''(x) = +6(1-x)^{-4}$ $\quad f'''(0) = 6$ $\quad \frac{6}{3!} = 1$

$1 + 1x + 1x^2 + 1x^3 + \cdots$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} 1x^n$$

ex. Find the Taylor series for the function
f(x)=lnx about x=1.

$f(x) = \ln x \qquad f(1) = 0 \qquad \frac{0}{0!} = 0 \leftarrow n=0$

$f'(x) = \frac{1}{x} = x^{-1} \qquad f'(1) = 1 \qquad \frac{1}{1!} = 1 \leftarrow n=1$

$f''(x) = -1x^{-2} \qquad f''(1) = -1 \qquad \frac{-1}{2!} = -\frac{1}{2}$

$f'''(x) = 2x^{-3} \qquad f'''(1) = 2 \qquad \frac{2}{3!} = \frac{1}{3}$

$f^{(4)}(x) = -6x^{-4} \qquad f^{(4)}(1) = -6 \qquad \frac{-6}{4!} \quad -\frac{1}{4}$

$0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots = \ln x = \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n}\right)(x-1)^n$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \dots$$

Thm. Plain English version:
A function equals its Taylor series iff its LaGrange remainder goes to 0.

Thm. Convergence of Taylor series
If a function f has derivatives of all orders in an open interval I centered at c, then the equality

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = f(x)$$

holds iff there exists a number z between x and c such that

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!} = 0$$

The remainder theorem in words:
The sum is at least as accurate as the first unused term.

So,

$$\left|f(x) - P_n(x)\right| \le \frac{M_{n+1} \cdot |x|^{n+1}}{(n+1)!} \qquad \text{where} \qquad \left|f^{(n+1)}(x)\right| \le M_{n+1}$$

ex. Suppose that f is a function such that
$f(1)=1$,
$f'(1)=2$, and
$f''(x)=(1+x^3)^{-1}$ for all $x>-1$ $f''(1)=\frac{1}{2}$

a. Estimate f(1.5) using a quadratic Taylor polynomial.

$$T_2(x) = 1 + \frac{2}{1!}(x-1)^1 + \frac{\frac{1}{2}}{2!}(x-1)^2$$

$$T_2(1.5) = 1 + 2(1.5-1)^1 + \frac{1}{4}(1.5-1)^2$$

$$= 1 + 2\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right)^2$$

$$= 1 + 1 + \frac{1}{16} = 2\frac{1}{16} \approx f(1.5)$$

b. Find an upper bound for the error made in this approximation.

$$f'''(x) = -1(1+x^3)^{-2}(3x^2)$$

max

$1 \le x < 1.5$ $\rightarrow$ $\dfrac{\left|\dfrac{-1(3x^2)}{(1+x^3)^2}\right|(x-1)^3}{3!} \le \dfrac{\left|\dfrac{-1(3)}{2^2}\right|(1.5-1)^3}{6}$

biggest here

$$= \frac{\frac{3}{4}\left(\frac{1}{2}\right)^3}{6} = \cancel{\frac{3}{4}}\cdot\frac{1}{8}\cdot\frac{1}{\cancel{6}_2}$$

$$= \frac{1}{64}$$

The movie the fugitive -- you know the one with Harrison Ford as the escaped prisoner and Tommy Lee Jones as the US Marshall trying to find him?

Well the first thing that Tommy Lee Jones does upon arriving at the escape scene is to perform a LaGrange error bound on the location of the escaped prisoner. Specifically, he knows where Harrison Ford was at the time he escaped (ie the value of the function at x=0) and he knows the maximum speed (velocity -- ie the first derivative) that the prisoner could achieve. So, he takes this maximum derivative value, multiplies by the difference in the time interval and established a perimeter -- ie a circumference that he is certain serves as a boundary for the location of the prisoner. Now, do we know exactly where the prisoner is at this given time? No, of course we don't. We are just confident that the prisoner must be somewhere inside this boundary. Of course if we miraculously knew the prisoner's exact average velocity on the time interval, then we could find his exact position. Since this value is not know - we settle for a maximum velocity on the interval.

Alright, let's take another example -- say a comet is moving through space and we just happen to stumble upon its existence while we were stargazing one night. Now, we are very interested in this comet and begin to record some things about it. How big it looks, what color it is and of course, where it is. We track it for a while and through the miracle of science are able to record its exact location and its exact velocity before going to bed for the night. The next evening, of course, we want to take another look at our new comet. Well, where should we look? First, we take its last known location, multiply its velocity by the time interval, add this projected change in position to the previous location and then look in that place in the night sky. The comet would be there - unless of course, it's velocity was not constant over the time interval. So, to account for variations in its position cause by changes in its velocity, we use what we believe might be the greatest magnitude of its change in velocity (or its acceleration) and then establish a bounded area where we might look for this comet. Once again, if we were able to find the exact average acceleration on the time interval we could determine the exact position of the comet.

so, there we have it, a conversational basis for the LaGrange error bound. You take all of the information you know to be true about your function at some moment - its position, velocity, acceleration, jerk, snap, crackle, pop, etc. -- and at some point you decide you have enough terms. To find the bound for the error created by not using an infinite number of terms, you must use the screw you guys, im going home! maximum value of the first omitted derivative on the interval in question. Use this value -- along with the rest of the Taylor polynomial -- to find a range of values, considering once again that if you knew the exact average value of the first omitted derivative on the interval that you could find the exact value of the function.

ex. Find the Maclaurin series for f(x)=cos(2x), using substitution into the power series for cos(x).

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n}}{(2n)!}$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

ex. Find the Maclaurin series for f(x)=xcos(x), using multiplication by the power series for cos(x).

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

$$x^1 \cdot x^{2n}$$