9.4 Direct Comparison Test

Thm. Direct Comparison Test
Let $0<a_n<b_n$ for all n beyond some value.

if the series of this term diverges,

if the series of this term converges,

then so does the other series.

ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{5+3^n}$

Compare to

$\sum \frac{1}{3^n} = \sum 1\left(\frac{1}{3}\right)^n$

$\frac{1}{5+3^n} < \frac{1}{3^n}$

convergent geometric series

so the given series converges as well.

ex. Determine the convergence of $$\sum_{n=1}^{\infty}\frac{1}{3n^2+2}$$

$$\frac{1}{3n^2+2} < \frac{1}{3n^2} < \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$

convergent p series

Thus, this series converges, too

ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{5^n}{3n-2}$

$$\frac{5^n}{3n-2} > \frac{5^n}{3n}$$

these terms do not go to 0
by the n^{th} term test,
this series $\sum_{n=1}^{\infty} \frac{5^n}{3n}$ diverges

so, $\sum_{n=1}^{\infty} \frac{5^n}{3n-2}$ will also diverge.

ex. Determine the convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$$

p series with $p = \frac{1}{2} \leq 1$
so it diverges

By the direct comparison test
the given series also diverges.

Thm. Limit Comparison Test

a_n and b_n are positive

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L$$

finite & positive

$\longrightarrow$

$\sum a_n$ & $\sum b_n$

both converge or both diverge.

Comparison series: keep only the highest powers of n in the numerator or denominator.

ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

compare to $\frac{n^2}{n^5} = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{\frac{2n^2-1}{3n^5+2n+1}}{\cancel{\frac{1}{n^3}}} \cdot \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1} = \frac{2}{3}$$ which is finite & positive

so, $\sum \frac{1}{n^3}$ converges and so does the given series.

ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{n+3}{n(n+4)}$

$\frac{n}{n^2} = \frac{1}{n}$

$\uparrow$

harmonic series

$$\lim_{n\to\infty} \frac{n+3}{\cancel{n}(n+4)} \cdot \frac{\cancel{n}}{1} = 1$$

so the given series diverges,
just as the harmonic series diverges

ex. Determine the convergence of $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

$\frac{1}{\sqrt{n^2}} = \frac{1}{n} \leftarrow$ harmonic series

$\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} \cdot \frac{\sqrt{n^2}}{1}$

$= \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2-1}} = 1$ so these series both diverge.