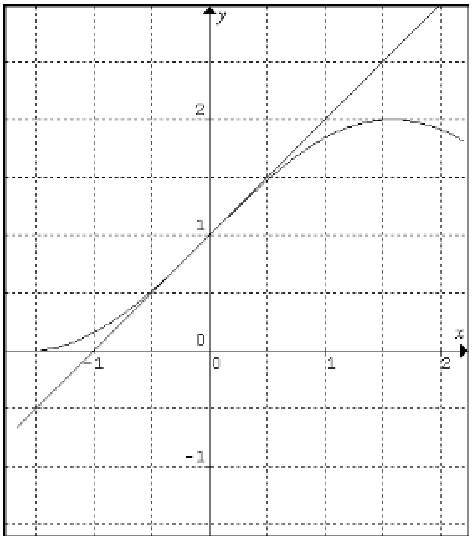
9.7 Taylor Polynomials and Approximations

Section 3.9, we approximated f(x)=1+sinx
with the line tangent at (0,1), y=x+1.



These have the
same point and
the same slope at that point.

How well can a polynomial
approximate f(x) near x=0?

Let's look at how a polynomial might approximate f(x)=1+sinx around x=0.

http://www.ma.utexas.edu/cgi-pub/kawasaki/plain/infSeries/janelas/taylor1.html

Notes on operation:
Change f to 1+sinx
change n to show number of terms
a tells where it is centered

http://www.slu.edu/classes/maymk/GeoGebra/TaylorPoly.html

Now look at how a polynomial might approximate $f(x)=\cos x$ around $x=\pi/4$.

http://sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Plot/calc.html

Notes on use:
change range of x to end at $\pi+1$
center graph at $x=\pi/4$ or .785398

These are other sites that could be used to see this:

http://math.furman.edu/~dcs/java/taylor.html

http://www.plu.edu/~heathdj/java/calc2/Taylor.html

<u>The big idea:</u>
at some x value, we want the polynomial and the original function share the same:

y value,
y' value,
y'' value,
y''' value...

... up to some desired degree of accuracy.

Some notation and vocabulary:

We usually call the original function f(x) and the approximating polynomial P(x) or T(x).

When looking near x=	0	c
the approximating polynomial is said to be centered at	0	c
or expanded about	0	c
the polynomial is also called a	Maclaurin polynomial	Taylor polynomial

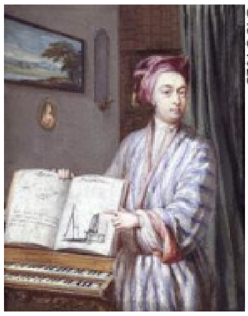
More terminology:

If P(x) and f(x) both have n derivatives that are equal,
P(x) is called either:

the nth Taylor polynomial for f at c,
an nth degree polynomial approximation for f at c, or
an nth order polynomial approximation for f at c,

Blame two English mathematicians:

Colin Maclaurin	Brook Taylor
1698-1746	1685-1731



This is the recipe for finding a Taylor Polynomial

Definition of nth Taylor Polynomial:

If f has n derivatives at c, then

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Remember, if c=0, we call it the nth Maclaurin polynomial.

ex. Find a 4th order approximation for $f(x) = e^x$ around x=0.

1. Find the coefficients:

f(x)=	f(0)=	÷ 0!=
f '(x)=	f '(0)=	÷ 1!=
f ''(x)=	f ''(0)=	÷ 2!=
f '''(x)=	f '''(0)=	÷ 3!=
f ''''(x)=	f ''''(0)=	÷ 4!=

2. Write the polynomial:

ex. Find a 3rd order approximation for $f(x)=\sin x$ around $x=\pi$.

1. Find the coefficients:

$f(x)=$	$f(\pi)=$	$\div\ 0!=$
$f'(x)=$	$f'(\pi)=$	$\div\ 1!=$
$f''(x)=$	$f''(\pi)=$	$\div\ 2!=$
$f'''(x)=$	$f'''(\pi)=$	$\div\ 3!=$

2. Write the polynomial:

Remember, it's only an approximation for a function!

$$f(x) = P_n(x) + R_n(x)$$

Exact function — Taylor Polynomial — Remainder

Consider the nth Taylor Polynomial around x=c.
For each x value around c, there is a number z between x and c such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

This is called the Lagrange form of the remainder.

How do we actually use that Lagrange Error Bound?

Suppose $f(x) = e^{5x}$ and

$$P_3(x) = 1 + 5(x-0) + \frac{25(x-0)^2}{2!} + \frac{125(x-0)^3}{3!}$$

is the 3rd degree Taylor polynomial for f about x=0. Use the Lagrange Error Bound to determine the absolute error between f(0.1) and P(0.1).

We start here:

$$\left|f(x) - P_3(x)\right| \leq \frac{\max\limits_{0\leq c\leq 0.1}\left|f^{(4)}(c)\right|}{4!}\left|0.1-0\right|^4$$

Notice that f is an increasing function, and so is each derivative. So, this max occurs when c=0.1 or the right endpoint of this interval.

$$= \frac{\left|625e^{0.1}\right|}{24}\left|0.0001\right|$$

$$= 0.002878$$

Use the Lagrange Error Bound to show that the 3rd degree Taylor polynomial for sin(x) about x=π approximates sin(3) with error less than 7×10^{-5}

$$f(x) = \sin(x)$$

$$\left|f(3) - P_3(3)\right| \le \frac{\max\limits_{3 \le c \le \pi}\left|f^{(4)}(c)\right|}{4!}\left|3 - \pi\right|^4$$

The 4th derivative of sin(x) is sin(x) and it decreases around π, so sin(3) would be the max of this derivative, but we'll use 1 for the max since we know that sin(x) never gets larger than 1.

$$\le \frac{|1|}{24}\left|3 - \pi\right|^4$$

$$< \frac{|1|}{24}\left|.2\right|^4$$

$$= 6.667 \times 10^{-5}$$

$$< 7 \times 10^{-5}$$