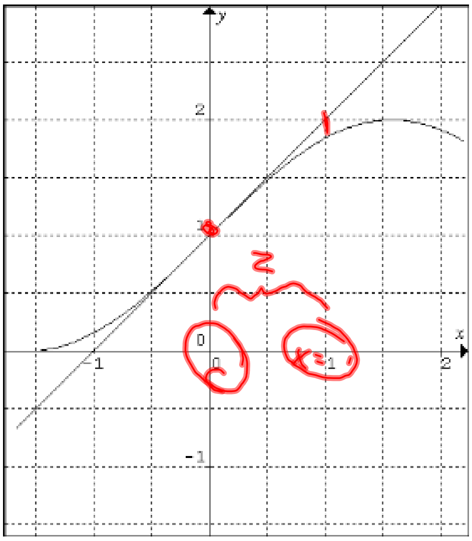
9.7 Taylor Polynomials and Approximations

$f''(x) = -\sin x$

Section 3.9, we approximated f(x)=1+sinx with the line tangent at (0,1), y=x+1.

$\frac{|-\sin(z)|}{2!}(1-0)^2$

$\frac{1}{2} \cdot 1$



These have the same point and the same slope at that point.

How well can a polynomial approximate f(x) near x=0?

Let's look at how a polynomial might approximate f(x)=1+sinx around x=0.

http://www.ma.utexas.edu/cgi-pub/kawasaki/plain/infSeries/janelas/taylor1.html

Notes on operation:
Change f to 1+sinx
change n to show number of terms
a tells where it is centered

http://www.slu.edu/classes/maymk/GeoGebra/TaylorPoly.html

Now look at how a polynomial might approximate f(x)=cosx around x=π/4.

http://sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Plot/calc.html

Notes on use:
change range of x to end at π+1
center graph at x=π/4 or .785398

These are other sites that could be used to see this:

http://math.furman.edu/~dcs/java/taylor.html

http://www.plu.edu/~heathdj/java/calc2/Taylor.html

The big idea:

at some x value, we want the polynomial and the original function share the same:

y value,
y' value,
y'' value,
y''' value...

... up to some desired degree of accuracy.

Some notation and vocabulary:

We usually call the original function f(x) and the approximating polynomial P(x) or T(x).

When looking near x=	0	c
the approximating polynomial is said to be centered at	0	c
or expanded about	0	c
the polynomial is also called a	Maclaurin polynomial	Taylor polynomial

More terminology:

If P(x) and f(x) both have n derivatives that are equal,
P(x) is called either:

the nth Taylor polynomial for f at c,
an nth degree polynomial approximation for f at c, or
an nth order polynomial approximation for f at c,

Blame two English mathematicians:

Colin Maclaurin 1698-1746

Brook Taylor 1685-1731

This is the recipe for finding a Taylor Polynomial

Definition of nth Taylor Polynomial:

If f has n derivatives at c, then

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Remember, if c=0, we call it the nth Maclaurin polynomial.

ex. Find a 4th order approximation for $f(x) = e^x$ around x=0. ← Maclaurin

1. Find the coefficients:

f(x)= e^x	f(0)= 1	÷ 0!= $\frac{1}{1} = 1$
f '(x)= e^x	f '(0)= 1	÷ 1!= $\frac{1}{1} = 1$
f ''(x)= e^x	f ''(0)= 1	÷ 2!= $\frac{1}{2} = \frac{1}{2}$
f '''(x)= e^x	f '''(0)= 1	÷ 3!= $\frac{1}{6} = \frac{1}{6}$
f ''''(x)= e^x	f ''''(0)= 1	÷ 4!= $\frac{1}{24} = \frac{1}{24}$

2. Write the polynomial:

$$1 + 1x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 = P_4(x)$$

$$P_4(x) \approx f(x) = e^x$$

ex. Find a 3rd order approximation for f(x)=sin x around x=π.

1. Find the coefficients:

f(x)= $\sin x$	f(π)= 0	÷ 0!= $\frac{0}{1} = 0$
f '(x)= $\cos x$	f '(π)= -1	÷ 1!= $\frac{-1}{1} = -1$
f ''(x)= $-\sin x$	f ''(π)= 0	÷ 2!= $\frac{0}{2} = 0$
f '''(x)= $-\cos x$	f '''(π)= 1	÷ 3!= $\frac{1}{6} = \frac{1}{6}$

2. Write the polynomial:

$$P_3(x) = 0 - 1(x-\pi)^1 + 0(x-\pi)^2 + \frac{1}{6}(x-\pi)^3$$

$$= -(x-\pi) + \frac{1}{6}(x-\pi)^3 \approx \sin x$$

Remember, it's only an approximation for a function!

$$f(x) = P_n(x) + R_n(x)$$

Exact function — Taylor Polynomial — Remainder

Consider the nth Taylor Polynomial around x=c.
For each x value around c, there is a number z between x and c such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

This is called the Lagrange form of the remainder.

How do we actually use that Lagrange Error Bound?

Suppose $f(x) = e^{5x}$ and

$$P_3(x) = 1 + 5(x-0) + \frac{25(x-0)^2}{2!} + \frac{125(x-0)^3}{3!}$$

is the 3rd degree Taylor polynomial for f about x=0. Use the Lagrange Error Bound to determine the absolute error between f(0.1) and P(0.1).

We start here:

$$\left|f(x) - P_3(x)\right| \le \frac{\max\limits_{0 \le c \le 0.1}\left|f^{(4)}(c)\right|}{4!}\left|0.1 - 0\right|^4$$

Notice that f is an increasing function, and so is each derivative. So, this max occurs when c=0.1 or the right endpoint of this interval.

$$= \frac{\left|625e^{\sout{0.1}\ .5}\right|}{24}\left|0.0001\right|$$

$$= \sout{0.002878}$$

.00429

Use the Lagrange Error Bound to show that the 3rd degree Taylor polynomial for sin(x) about x=π approximates sin(3) with error less than 7×10^{-5}

$$f(x) = \sin(x)$$

$$\left|f(3) - P_3(3)\right| \le \frac{\max\limits_{3 \le c \le \pi}\left|f^{(4)}(c)\right|}{4!}\left|3 - \pi\right|^4$$

The 4th derivative of sin(x) is sin(x) and it decreases around π, so sin(3) would be the max of this derivative, but we'll use 1 for the max since we know that sin(x) never gets larger than 1.

$$\le \frac{|1|}{24}\left|3 - \pi\right|^4$$

$$< \frac{|1|}{24}\left|.2\right|^4$$

$$= 6.667 \times 10^{-5}$$

$$< 7 \times 10^{-5}$$