9.8 Power Series

Write a geometric series with x instead of r and you get:

Defn. $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots$

is called a <u>power series</u> &

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c)^1 + a_2 (x-c)^2 + a_3 (x-c)^3 + \ldots$$

is called a <u>power series centered at c</u> for some constant c.

Remember:

$0<|r|<1$ brings convergence & $|r|\geq 1$ brings divergence

so, $0<|x|<1$ brings convergence & $|x|\geq 1$ brings divergence

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

ex. Based on what we did with geometric series, what does the power series

$$\sum_{n=0}^{\infty} 1x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

converge to?

$a = 1$

$r = x$

$$f(x) = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

$0 < |x| < 1$

Thm. Three cases:
A power series centered at c, either converges...

A. for c only (only at $x=c$).	B. absolutely for all x when $\|x-c\| < R$, and diverges otherwise.	C. for all x.

The number R is called the <u>radius of convergence</u> of the power series.

A. $R=0$.	B. R is some real number.	C. $R=\infty$

The set of all x values for which the power series converges is called the <u>interval of convergence</u>.

To find the interval of convergence

1. Ratio test or root test finds interval of absolute convergence, usually $|x-c| < R$.
2. If case B, use some test for convergence at the endpoints.

ex. Determine the interval of convergence of $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)$

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{\cancel{n+1}}}{\underset{4\cdot\cancel{3\cdot 2\cdot 1}}{(n+1)!}}\cdot\frac{\overset{\cancel{3\cdot 2\cdot 1}}{n!}}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x|$$

$$= 0\cdot|x| = 0 < 1$$

this converges absolutely

$(-\infty, \infty)$

case C

ex. Determine the interval of convergence of $\sum_{n=0}^{\infty} n!x^n$

x^n

$\lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty} \left|(n+1)x\right|$

$(x-0)^n$

c

$= \lim_{n\to\infty} (n+1)|x|$

∞

Converges only at 0

case A

[0, 0]

ex. Find the interval of convergence of $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{x^n}{n}\right)$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot |x| = 1 \cdot |x| < 1 \text{ to converge}$$

$$(-1, 1]$$

check $x = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot(-1)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n}$$

diverges, since it's negative harmonic

Check $x = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot 1^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges, since it's an alternating harmonic ex. Determine the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$

$$\lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}}{(n+1)\sqrt{n+1}\;\cancel{3^{n+1}}}\cdot\frac{n\sqrt{n}\;\cancel{3^n}}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\frac{n\sqrt{n}}{(n+1)\sqrt{n+1}\cdot 3}|x| = \frac{1}{3}|x| < 1$$

so $|x| < 3$ $[-3, 3]$

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}\;3^n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}\cdot\left(\frac{-3}{3}\right)^n$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$$

alternating p series

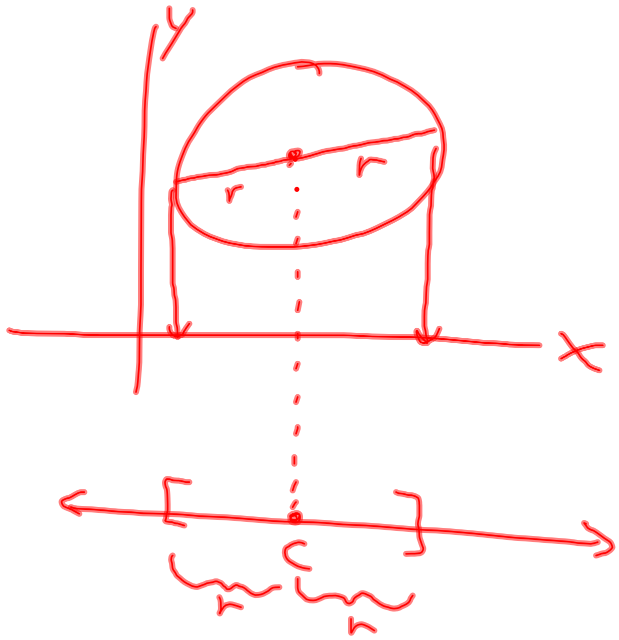
$p = \frac{3}{2}$

convergent

$$\sum_{n=1}^{\infty}\frac{\cancel{3^n}}{n\sqrt{n}\;\cancel{3^n}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$$

convergent p series


y
r
r
x
c
r
r

Thm. Power series can be integrated and differentiated term by term:

If a function $f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n$

has a radius of convergence R>0, then on (c-R,c+R) f is continuous, differentiable, and integrable and in fact,

1. $f'(x)=\sum_{n=0}^{\infty} na_n(x-c)^{n-1}$ and
2. $\int f(x)dx=C+\sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1}$

with all three series having the same R, but possibly different intervals due to behavior at the endpoints.

ex. Use the fact that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$$

converges for all x to

a. Write the first 6 terms of a series for cosx.

b. Write a series for sin(2x).

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \frac{11x^{10}}{11!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

sinx

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} =$$

b. $$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$$