Parametric Equations, Vectors, and Vector Valued Functions

Different parametric equations can yield the same curve:

$x=t$, $y=t^2$ for t in [-1,1] and $x=t^3$, $y=t^6$ for t in [-1,1]

give the same parabolic arc,
but look at where t=-1, 0, 1, and 0.75 to see the difference.

(-1,1)
(0,0)
(1,1)
(0.75, 0.5625)

(-1, 1)
(0, 0)
(1, 1)
(.421875, .17797...)

Speed for parametrics = $\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$

Arc length for parametrics = ∫speed dt = $\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$

(consider the area under the graph of speed vs. time and see that distance/unit time * time = distance traveled)

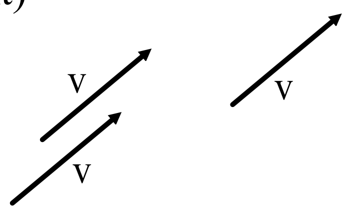
A vector is a quantity with both magnitude and direction.

In a plane, a vector, v, is thought of as:

- an arrow (any 2 arrows with the same magnitude and direction are equivalent, regardless of placement)

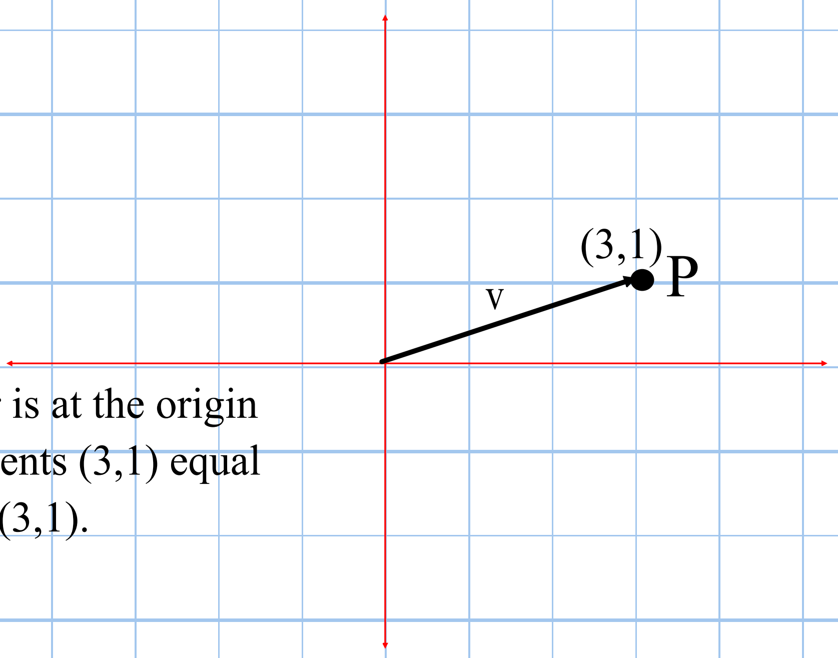
or

- pair of components or coordinates, v=(a, b).



position vector :

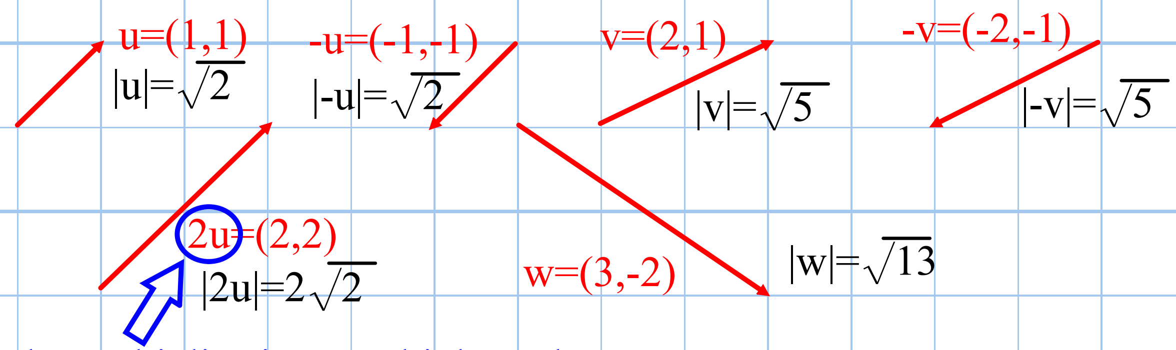
- the tail of the vector is at the origin
- the vector's components equal point P's coordinates.



ex. position vector v:

- the tail of the vector is at the origin
- the vector's components (3,1) equal point P's coordinates (3,1).

The components of the vector show the displacement between the vector's head and tail.



Scalar multiplication: multiply each component by a scalar (constant). A scalar factor r stretches a vector |r| units.

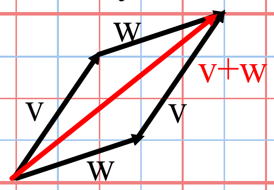
The length or magnitude of a vector = $|v| = \sqrt{(a)^2+(b)^2}$

(Recall speed for parametrics. It's the same idea.)

A unit vector has length 1 unit.

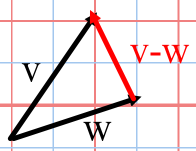
Sum of vectors: just add the corresponding components. The sum of vectors follows the parallelogram rule: the sum v+w is the diagonal of the parallelogram formed by v and w.

(2,3)+(3,1)=(5,4)



Difference of vectors: just subtract the corresponding components. The difference of vectors follows the tip triangle rule: the difference v-w is the vector from the tip of w to the tip of v, forming a triangle.

(2,3)-(3,1)=(-1,2)



Note about some books, courses, authors, or instructors:
Sometimes we denote vectors in terms of <u>standard basis vectors</u>, $i=(1,0)$ and $j=(0,1)$.

ex. $v=(3,4) = 3(1,0) + 4(0,1) = 3i+4j$

Any vector $(a,b)=ai+bj$

<u>Vector valued function</u>: input is one variable and output is vectors. (Some books, courses, authors, or instructors use a boldface ***f*** to denote vector valued functions and plain *f* for functions that map a real number to a real number.)

Ex. In the vector valued function ***f***(t)=(cos(t),sin(t)) , the functions in the parentheses are called <u>component functions</u> or <u>coordinate functions</u>. This sends a real number t to a vector (cos t , sin t).

Domain Range

$\mathbb{R} \longrightarrow$ vectors

x y

$f(0) = (\cos 0, \sin 0) = (1, 0)$

or $i + 0j$ or just i

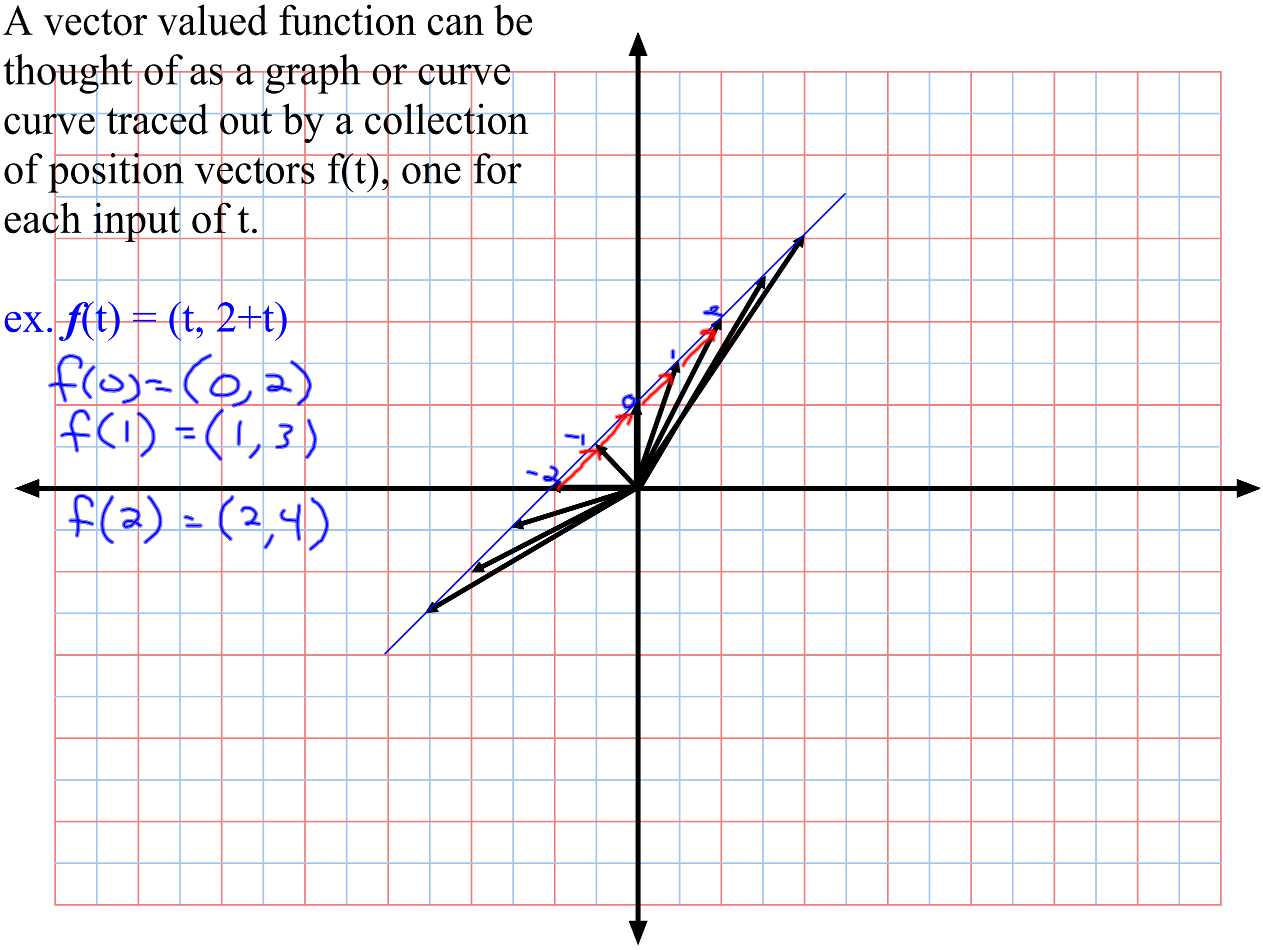
A vector valued function can be thought of as a graph or curve curve traced out by a collection of position vectors f(t), one for each input of t.

ex. $\boldsymbol{f}(t) = (t, 2+t)$

$f(0) = (0,2)$

$f(1) = (1,3)$

$f(2) = (2,4)$



Find the derivative of a vector valued function, by differentiating each component.

$\boldsymbol{f}(t) = (f_1(t)\ ,\ f_2(t))$ has derivative $\boldsymbol{f}'(t) = (f_1'(t)\ ,\ f_2'(t))$.

If $\boldsymbol{f}(t)$ is position at time t, then $\boldsymbol{f}'(t)$ is velocity at that time.

The velocity and acceleration are both vectors. The velocity vector tangent to the position curve at a given t value points in the direction of increasing t and has a magnitude that tells the instantaneous speed.

Also find antiderivatives and integrals one component at a time.

ex. Find the velocity of each position vector.

(cost, sint) (t, 2+t)

$v(t) = (-\sin t, \cos t)$ $v(t) = (1, 1)$

The arc length for a vector valued position function follwos the same rules as for parametrics.

That rule again is:

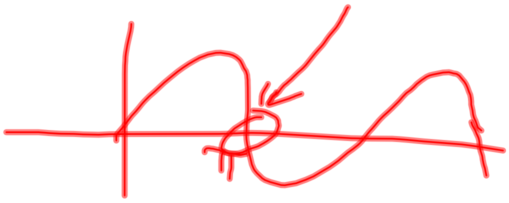
Arc length for parametrics
(or a vector valued position function) = ∫speed dt = $\int_{a}^{b}\sqrt{\left(x'(t)\right)^{2}+\left(y'(t)\right)^{2}}\,dt$

ex. A roller coaster travels a path determined by
$x(t)=10+10t-20\sin t$ $=84$ $x'(t)=10-20\cos t$ $x''(t)=20\sin t$
$y(t)=30-20\cos t$ $y'(t)=20\sin t$ $y''(t)=20\cos t$
over the interval $0\leq t\leq 10$, with t measured in seconds and both x and y are measured in meters.

a. Find the slope of the path at time t=2. $\frac{y'(2)}{x'(2)}=1$
b. Find the acceleration vector of the car when its horizontal position is x=84. $(13.104, -15.109)$
c. Find the time at which the car first reaches its maximum height, find the velocity vector of the car at this time, and find the speed, in m/sec, of the car at this time.
d. How far does the car travel in the interval $1\leq t\leq 4$? 45
e. For $0<t<4$, there are two times at which the car is 5 meters above the ground. Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

c.

$t = \pi$ 1st reaches max



$y(\pi) = 50$

velocity at $t = \pi$

$(x'(\pi), y'(\pi))$

$= (30, 0)$

$\langle 30, 0 \rangle$

Speed $= \|(30, 0)\| = \sqrt{30^2 + 0^2}$

$= 30$ m/sec

d arc length $= \int_1^4 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$$= \int_1^4 \sqrt{(10-20\cos t)^2 + (20\sin t)^2}\,dt$$

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fnInt(√((10-20cos(T))²+(20sin(T))²),T,1,4)
                    79.27667908
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meters

e

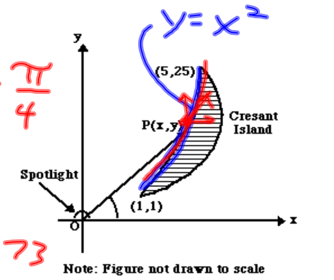
$$\frac{1}{3.864-2.419}\int_{2.419}^{3.864} \sqrt{(10-20\cos t)^2 + (20\sin t)^2}\,dt$$

ex. The figure below shows a spotlight shining on point P (x,y) on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point (1,1) to the point (5,25). Let θ be the angle between the beam of light and the positive x-axis.

(a) For what values of θ between 0 and 2π does the spotlight shine on the shoreline?

(b) Find the x- and y-coordinates of point P in terms of tan θ.

(c) If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point (3,9)?



Note: Figure not drawn to scale

a. $\tan\theta = \frac{y}{x} = \frac{1}{1} \rightarrow \theta = \frac{\pi}{4}$

$\tan\theta = \frac{25}{5} = 5$

$\theta = \tan^{-1} 5 = 1.373$

$\left[\frac{\pi}{4}, 1.373\right]$

$\cancel{x = r\cos\theta}$

$\cancel{y = r\sin\theta}$

$\cancel{r\sin\theta = (r\cos\theta)^2}$

b. $\boxed{\tan\theta = \frac{y}{x}} = \frac{x^2}{x}$

$\tan\theta = x$

$y = x^2 = \tan^2\theta$

$(\tan\theta, \tan^2\theta)$

c. speed $= \sqrt{\left(\sec^2\theta \cdot \frac{d\theta}{dt}\right)^2 + \left(2\tan\theta\sec^2\theta\,\frac{d\theta}{dt}\right)^2}$

$\frac{d\theta}{dt} = \frac{2\pi \text{ radians}}{\text{min}}$

$\sqrt{90}$, (3,9), 9, 3, θ

$= \sqrt{\left(\left(\frac{\sqrt{90}}{3}\right)^2 \cdot 2\pi\right)^2 + \left(2 \cdot \frac{\cancel{9}^{3}}{\cancel{3}_{1}} \cdot \left(\frac{\sqrt{90}}{3}\right)^2 \cdot 2\pi\right)^2}$

$= \sqrt{(10 \cdot 2\pi)^2 + (6 \cdot 10 \cdot 2\pi)^2}$

$= \sqrt{(20\pi)^2 + (120\pi)^2}$

$= \sqrt{20^2\pi^2 + 120^2 \cdot \pi^2}$

$= \sqrt{\pi^2(20^2 + 120^2)}$

$\pi \cdot 121.655$

382.191

$\tan^2\theta$

$= (\tan\theta)^2$

$2\tan\theta\sec^2\theta$