Algebra 2

# 3.7 The Fundamental Theorem of Algebra

Objectives:

- Use the Fundamental Theorem of Algebra to determine the number of solutions.
- Use Descartes Rule of Signs to identify possible roots of a polynomial equation.

## Core Concept

### The Fundamental Theorem of Algebra

**Theorem** If $f(x)$ is a polynomial of degree $n$ where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

**Corollary** If $f(x)$ is a polynomial of degree $n$ where $n > 0$, then the equation $f(x) = 0$ has exactly $n$ solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

## Descartes's Rule of Signs

French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

## Core Concept

### Descartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of $f$ is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of $f$ is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros.

deg = 6

Ex. 1: $f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8$

1 2 3

$f(-x) = x^6 + 2x^5 + 3x^4 + 10x^3 - 6x^2 + 8x - 8$

1 2 3

How many possible solutions?

| P<br>Positive | N<br>Negative | I<br>Imaginary | Total |
|---|---|---|---|
| 3 | 3 | 0 | 6 |
| 1 | 1 | 4 | 6 |
| 3 | 1 | 2 | 6 |
| 1 | 3 | 2 | 6 |

degree = 5

Ex. 2: $f(x) = 2x^5 - x^4 - 5x^3 + 3x^2 - 7x + 9$

1 2 3 4

$f(-x) = -2x^5 - x^4 + 5x^3 + 3x^2 + 7x + 9$

How many possible solutions?

1

| P | N | I | total |
|---|---|---|---|
| 4 | 1 | 0 | 5 |
| 2 | 1 | 2 | 5 |
| 0 | 1 | 4 | 5 |

## Finding All Roots of a Polynomial

**Step 1:** Use the Rational Root Theorem (p's and q's) to identify rational roots.
**Step 2:** Use Descartes's Rule of Signs to determine the possible number of positive, negative and imaginary solutions.
**Step 3:** Test the possible real roots using Synthetic Division.
**Step 4:** Use Synthetic Division and the depressed polynomial to find real roots, until the polynomial is reduced to a quadratic.
**Step 5:** Solve the quadratic by factoring, square roots, completing the square or the quadratic formula to find the remaining roots.

q: 1  p: 1, 2, 4

Ex. 3: Solve $x^3 - 4x^2 + 6x - 4 = 0$.

$\frac{p}{q}: \pm 1, \pm 2, \pm 4$

| P | N | I |
|---|---|---|
| 3 | 0 | |
| 1 | 0 | |

$-x^3 - 4x^2 - 6x - 4$

1 | 1  -4  6  -4
   1  -3  3
1  -3  3

2 | $1x^3$  -4  6  -4
   2  -4  4
$1x^2$  -2  2  0

$x^2 - 2x + 2$

$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$= \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2} = 1 \pm i$

$x = 2$
$x = 1 \pm i$

q: 1  p: 1, 2, 3, 4, 6, 9, 12, 18, 36

Ex. 4: Solve $x^4 - 3x^3 + 5x^2 - 27x - 36 = 0$.

$\frac{p}{q}: \pm$ of these

$x^4 + 3x^3 + 5x^2 + 27x - 36$

| P | N | I | degree |
|---|---|---|---|
| 3 | 1 | 0 | 4 |
| 1 | 1 | 2 | 4 |

-1 | $1x^4$  -3  5  -27  -36
   -1  4  -9  36
4 | $1x^3$  -4  9  -36  0
   4  0  36
$1x^2$  0  9  0

$x^2 + 9 = 0$
$-9 \quad -9$

$\sqrt{x^2} = \sqrt{-9}$

$x = \pm 3i$

$+3i, -3i, 4, -1$

$q: 1$

$p: 1, 2, 4, 8, 16$

Ex. 5: Solve $x^5 - x^4 - 2x^3 + 8x^2 - 8x - 16 = 0$.

$\frac{p}{q}$: $\pm$ of these

$-x^5 - x^4 + 2x^3 + 8x^2 + 8x - 16$

| P | N | I | deg |
|---|---|---|---|
| 3 | 2 | 0 | = 5 |
| 1 | 0 | 4 | = 5 |
| 3 | 0 | 2 | = 5 |
| 1 | 2 | 2 | = 5 |

$$\begin{array}{r|rrrrrr} 2 & 1 & -1 & -2 & 8 & -8 & -16 \\ & & 2 & 2 & 0 & 16 & 16 \\ \hline & 1 & 1 & 0 & 8 & 8 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & 0 & 8 & 8 \\ & & -1 & 0 & 0 & -8 \\ \hline & 1 & 0 & 0 & 8 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

$x^2 - 2x + 4 \rightarrow$ quad. formula

$$x = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$2, -2, -1, 1 \pm i\sqrt{3}$